Boat or Bullet: Prior Parameter Set Shapes and Posterior Imprecision

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- **prior-data conflict:** if $P(\text{win})$ is actually very different from our prior guess (prior information and data are in conflict), this should show up in the predictive inferences (probability P and, e.g., credibility intervals)

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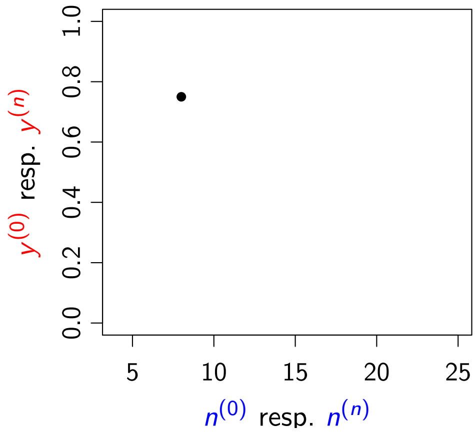

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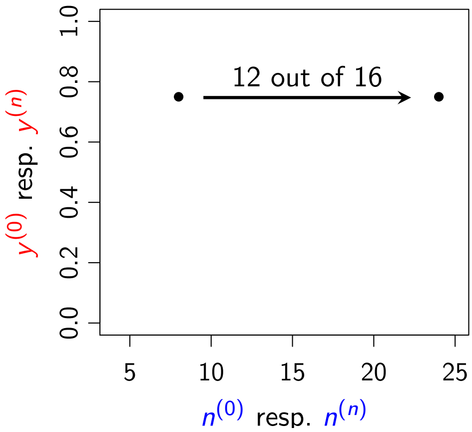

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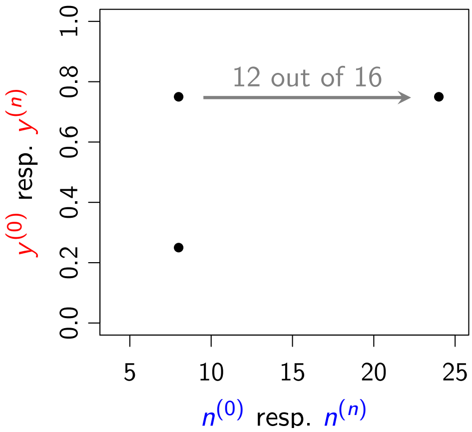

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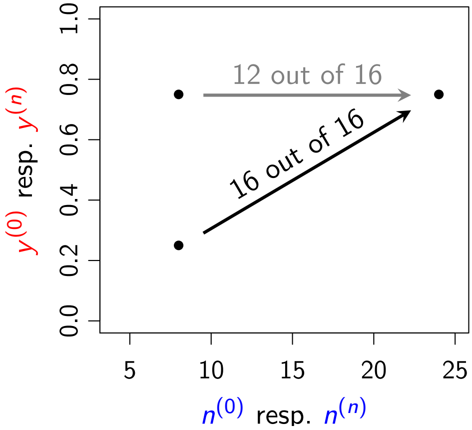

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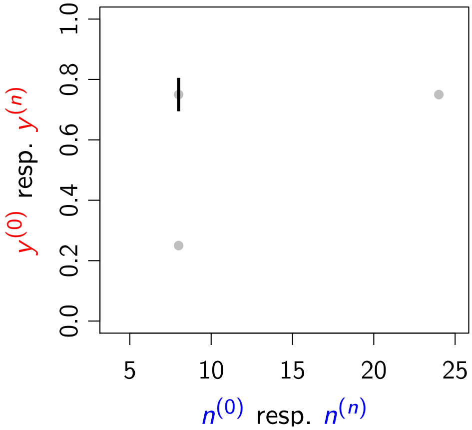

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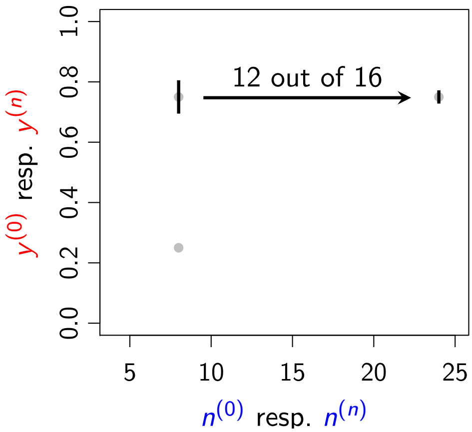

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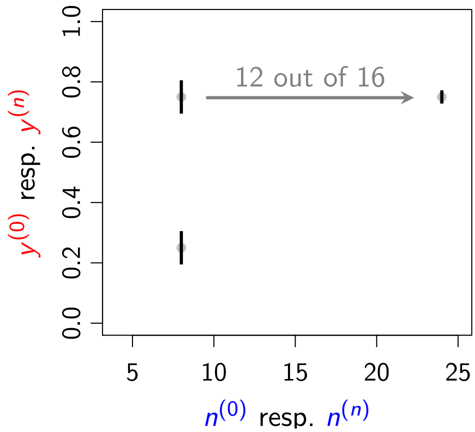

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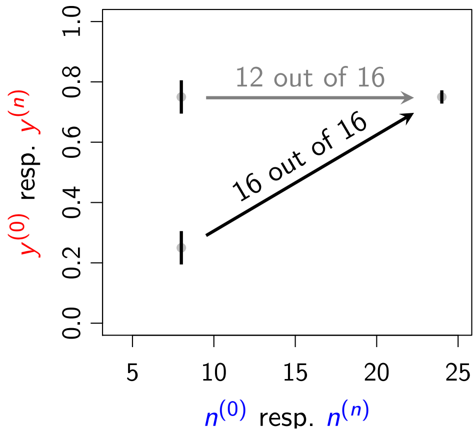

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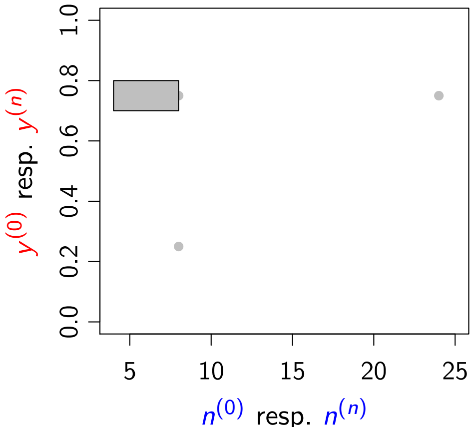

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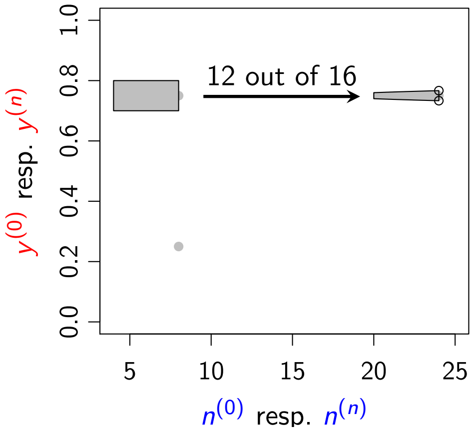

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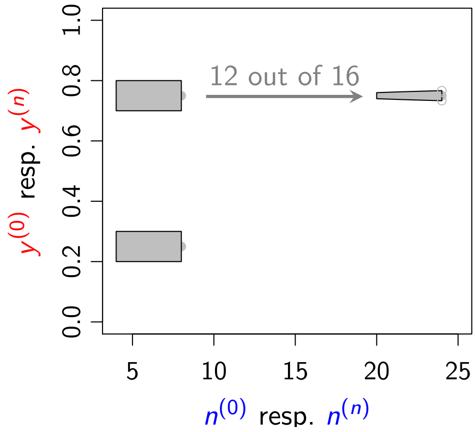

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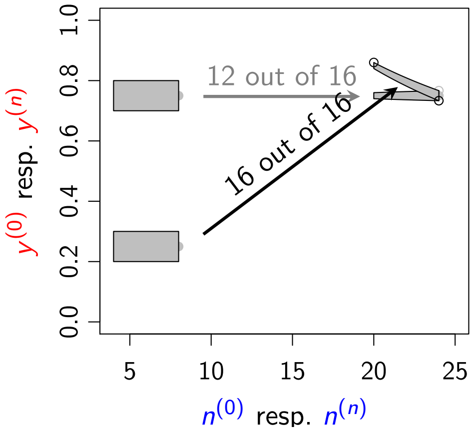

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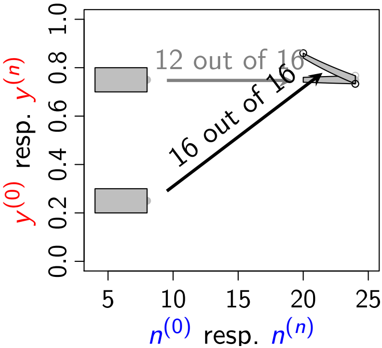

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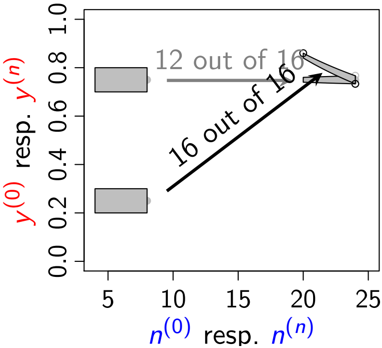

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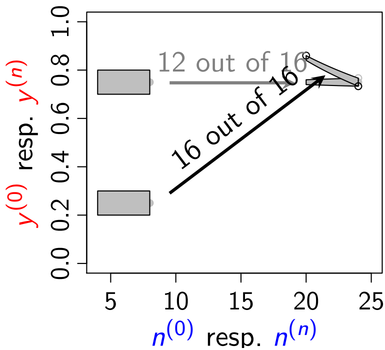

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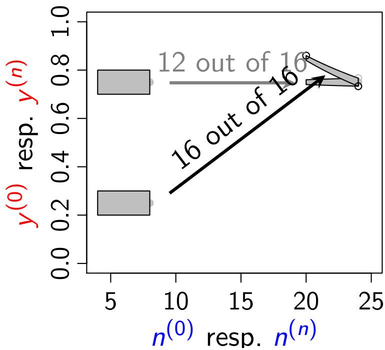

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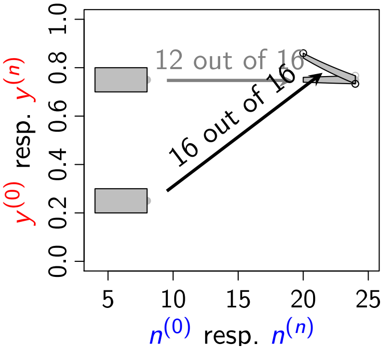

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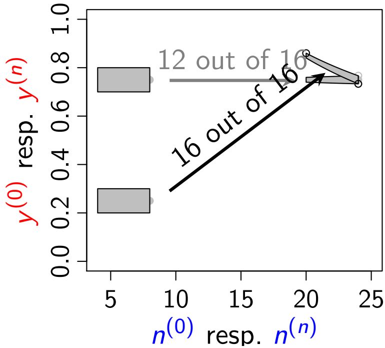

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- contrary effect of $n^{(0)}$ on imprecision and variance: high $n^{(0)}$ gives a highly imprecise posterior set, but with low variance distributions within the set

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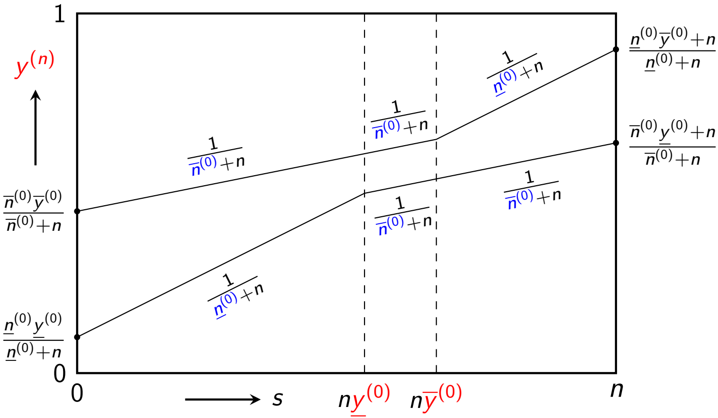

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- actual prior shape influences the posterior inferences (position of max / min $y^{(n)}$, other objective functions)

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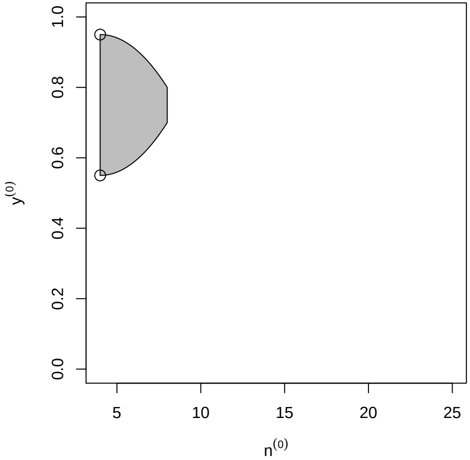

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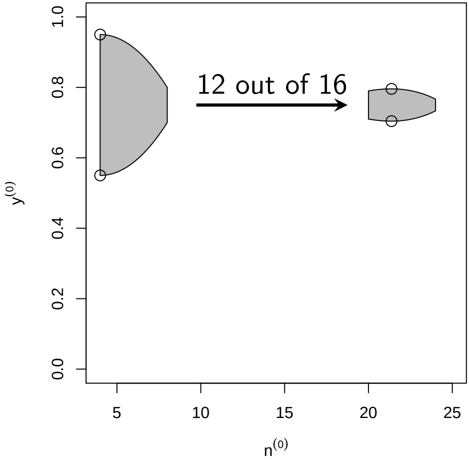

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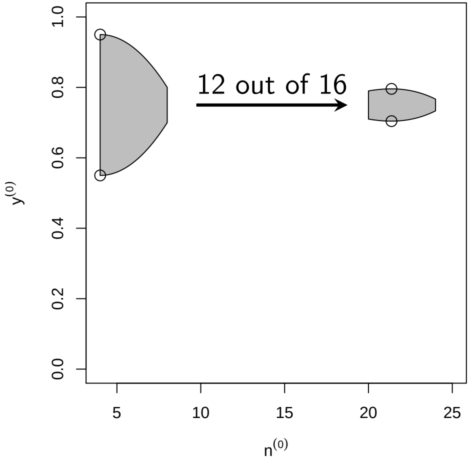

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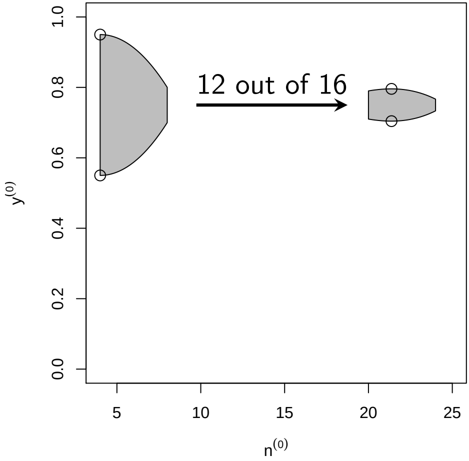

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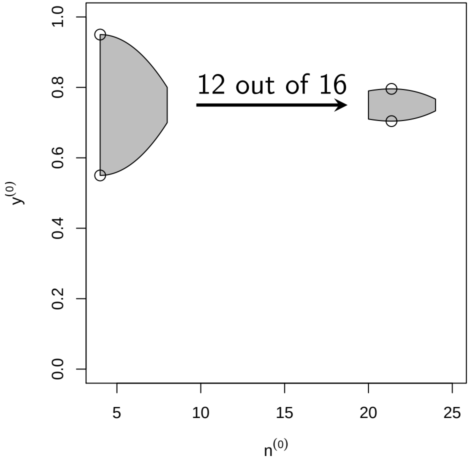

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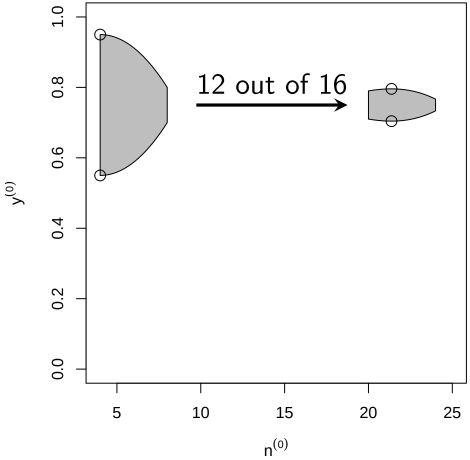

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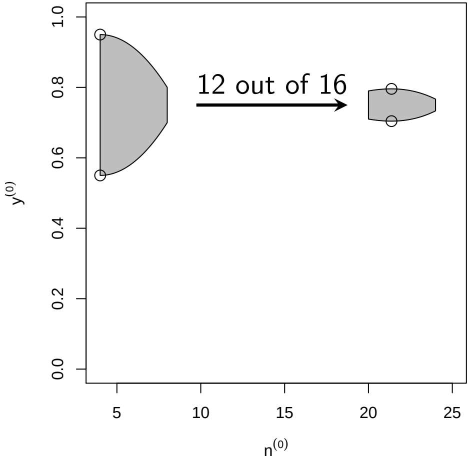

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- shape that has easy description for posterior set? (set description that's invariant under updating?)

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  (small variances!)
- deus ex machina: Mik Bickis & his approach to imprec. BBM:
  change parametrization $(n^{(0)}, y^{(0)})$ to (η_0, η_1) (WPMSIIP'11)
  ➡ the parameter sets do not change shape during updating!

Mik's Parametrization

- $(\eta_0, \eta_1) \in \mathbb{R}^2$ such that $\eta_0 > 2$, $-1 - \frac{1}{2}\eta_0 < \eta_1 < 1 + \frac{1}{2}\eta_0$
- $\eta_0 = n^{(0)} - 2$
- Updating: win $\hat{=} + \left(1, \frac{1}{2}\right)$, not win $\hat{=} + \left(1, -\frac{1}{2}\right)$
- Distributions with the same expectation ($=$ P) lie along equidistant rays emanating from $(-2, 0)$

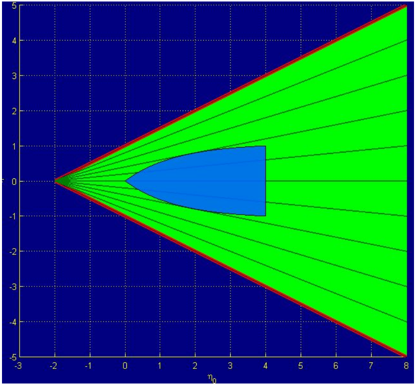
The Boat Shape

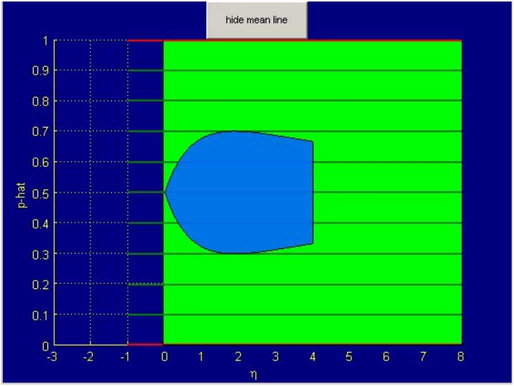
Our suggestion for a (η_0, η_1) shape that leads to

- additional imprecision in case of prior-data conflict
- bonus precision in case of strong prior-data agreement

looks like a bullet, or a boat with a so-called transom stern.

The Boat Shape





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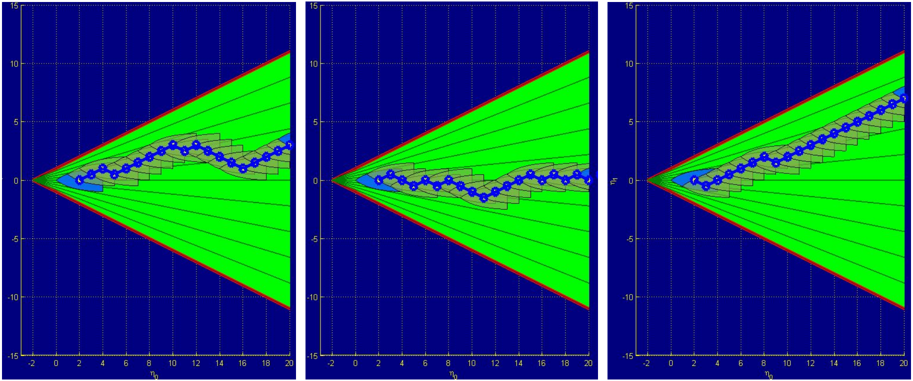

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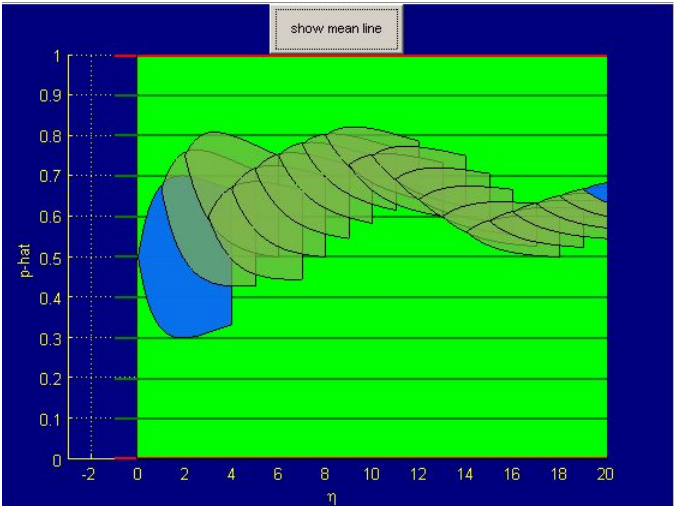

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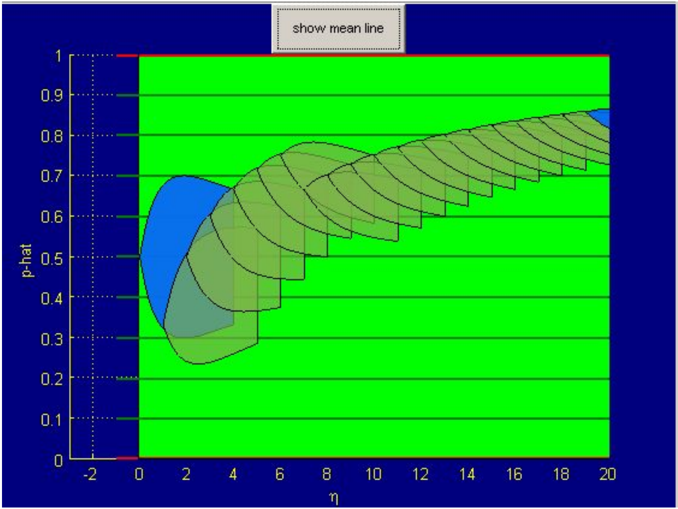

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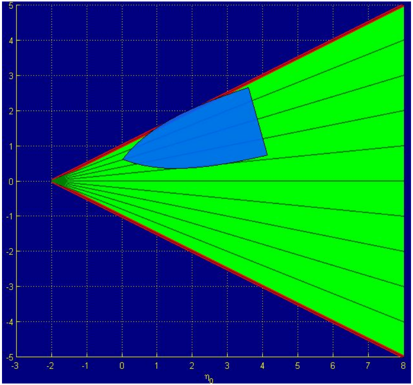
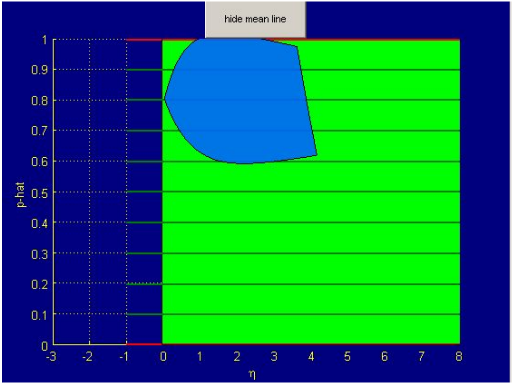
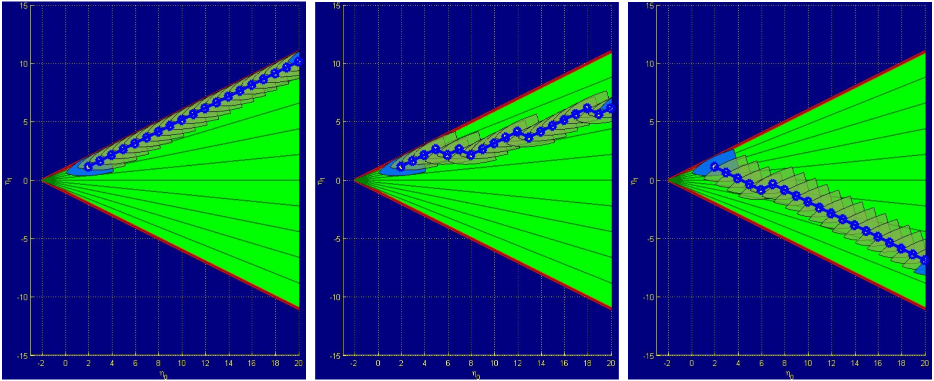
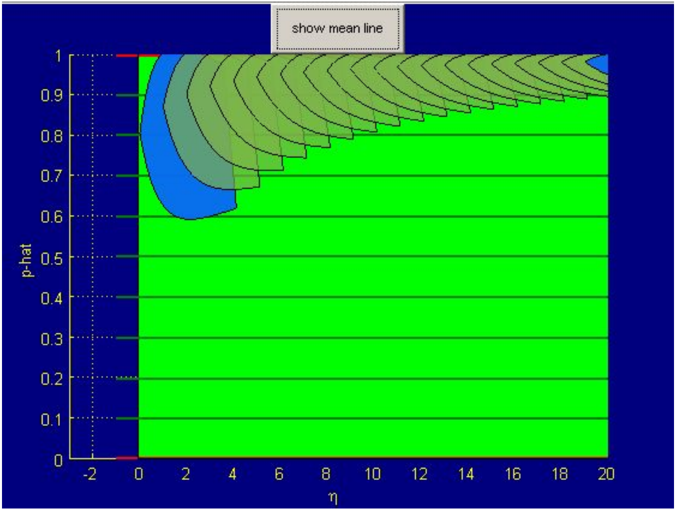




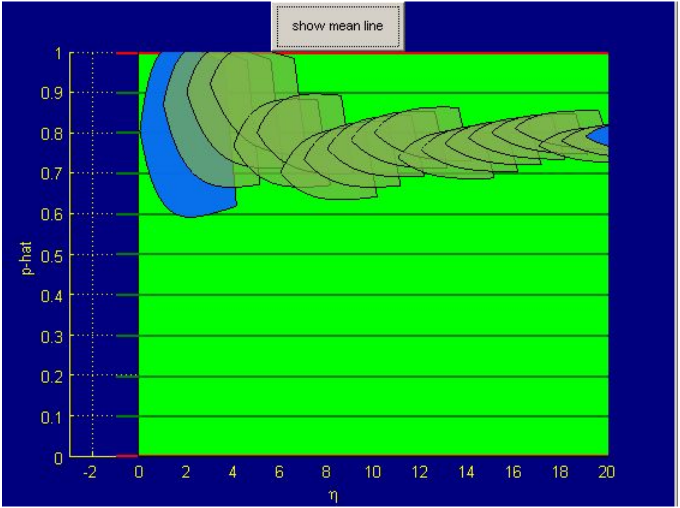
The Boat Shape



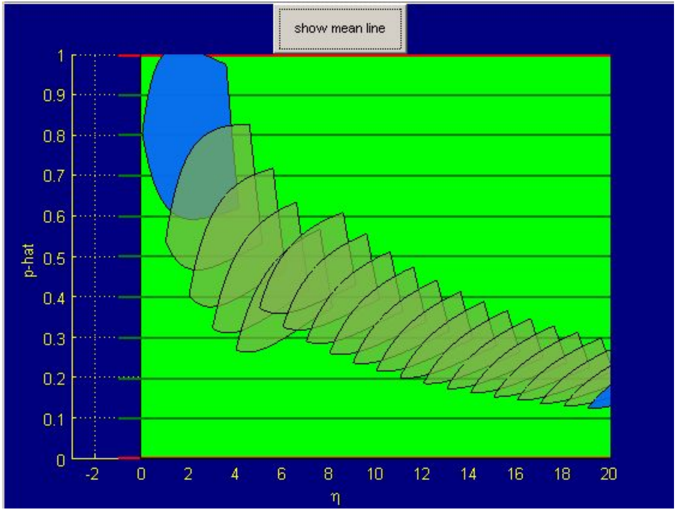
The Boat Shape



The Boat Shape



The Boat Shape



The Boat Shape

- interactive graph by Mik Bickis
- contours are exponential curves (mirrored at central ray)
- "touching rays" / "shadow" must be determined numerically
- detailed properties still to be explored
- suggestions?