Robust Bayesian Estimation of System Reliability with Scarce and Surprising Data

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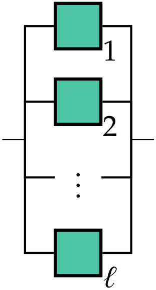
DINALOG Dutch Institute for Advanced Logistics

ESREL 2015

Setting: a one of a kind parallel system



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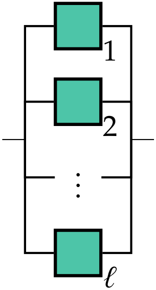

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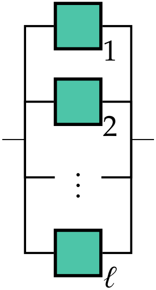

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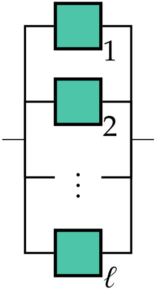

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How to combine these two information sources?

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- makes learning about component reliability tractable, just update parameters: $\alpha^{(0)} \rightarrow \alpha^{(\ell)}$, $\beta^{(0)} \rightarrow \beta^{(\ell)}$
- conjugacy holds also for censored observations
- closed form for system reliability function $R_{\mathsf{sys}}(t \mid \mathbf{t})$

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What if expert information and data tell different stories?

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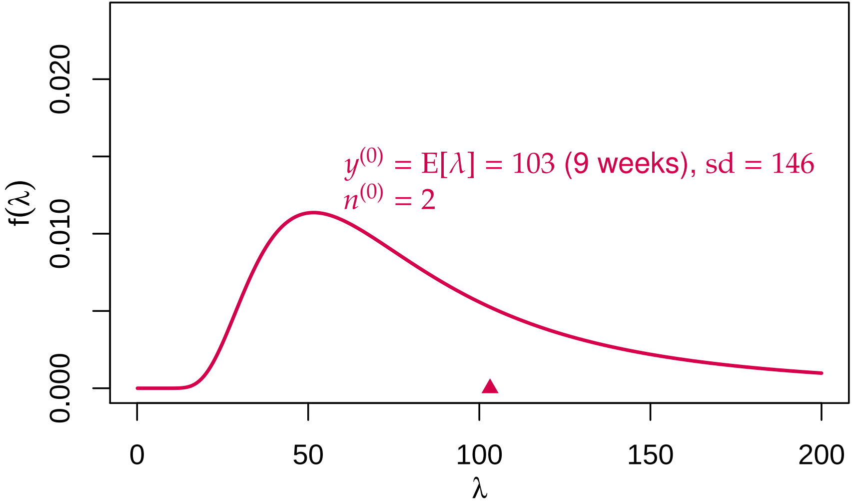
$\mathrm{E}[\lambda \mid \mathbf{t}]$ is a weighted average of $\mathrm{E}[\lambda]$ and $\hat{\lambda}$!

Prior-data conflict example



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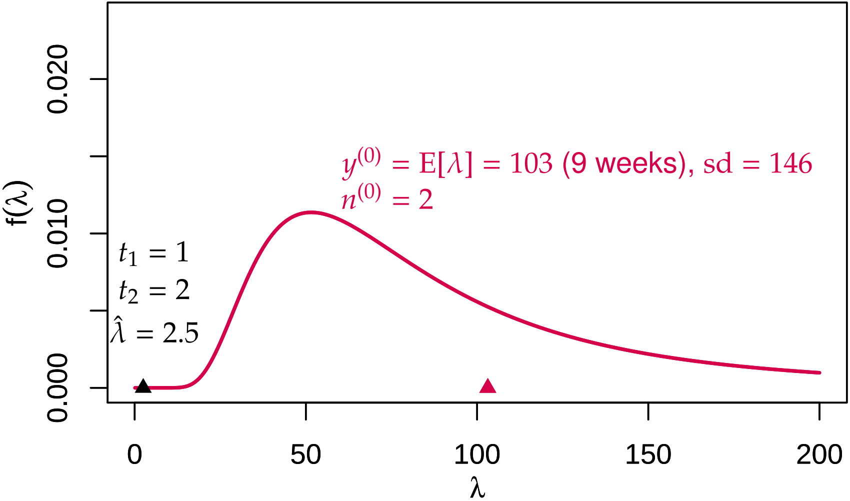

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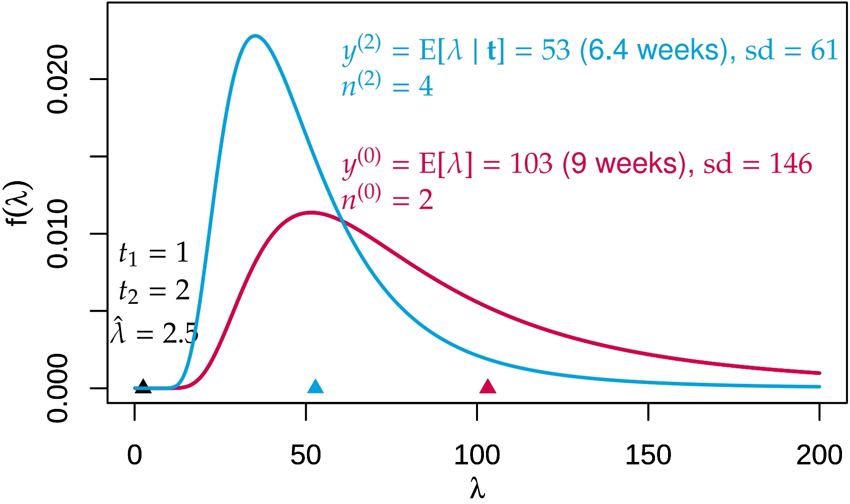

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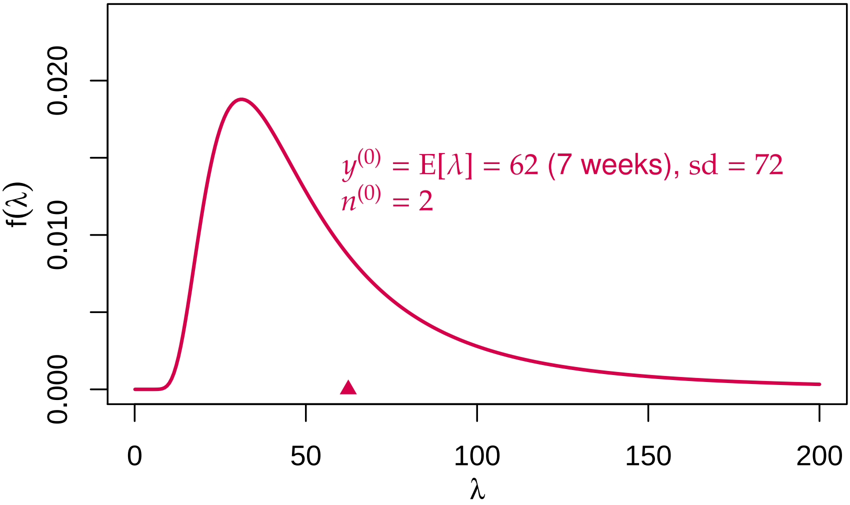

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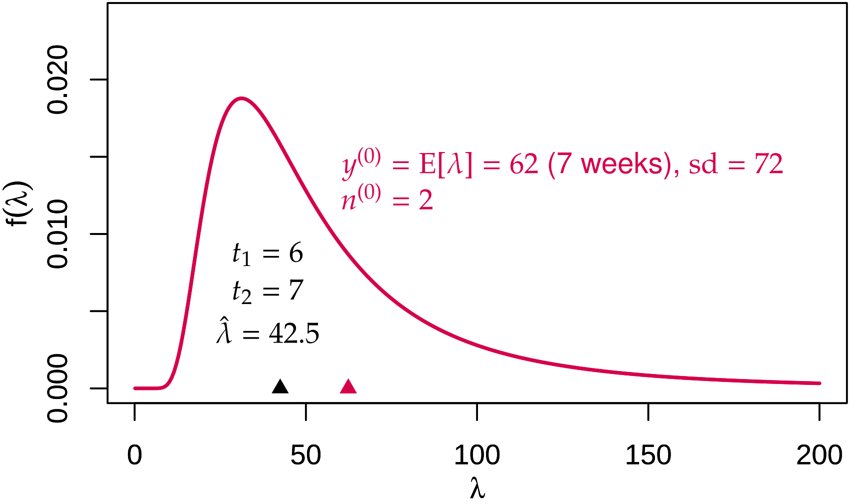

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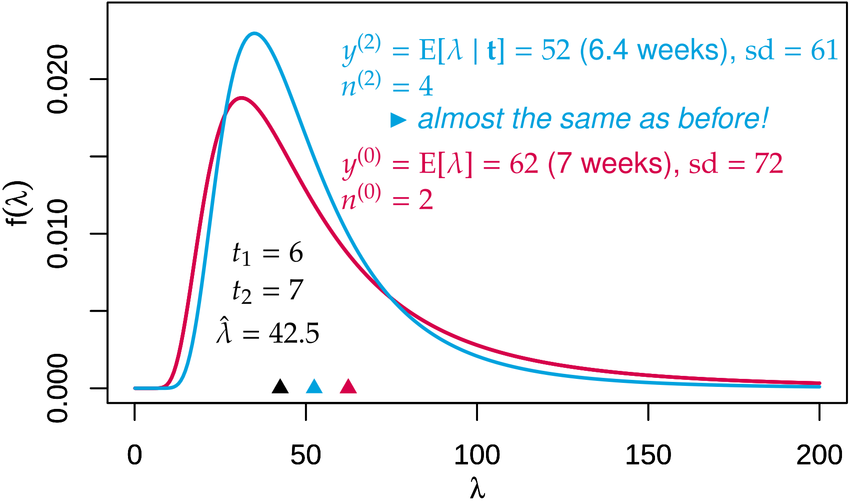

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- Separate uncertainty *whithin the model* (reliability statements)
  from uncertainty *about the model* (which parameters).
- Can also be seen as systematic sensitivity analysis
  or robust Bayesian approach.

Sets of prior distributions

Uncertainty about probability statements

smaller sets = more precise probability statements

Lottery A

Number of winning tickets:
exactly known as 5 out of 100

➡ $P(\text{win}) = 5/100$

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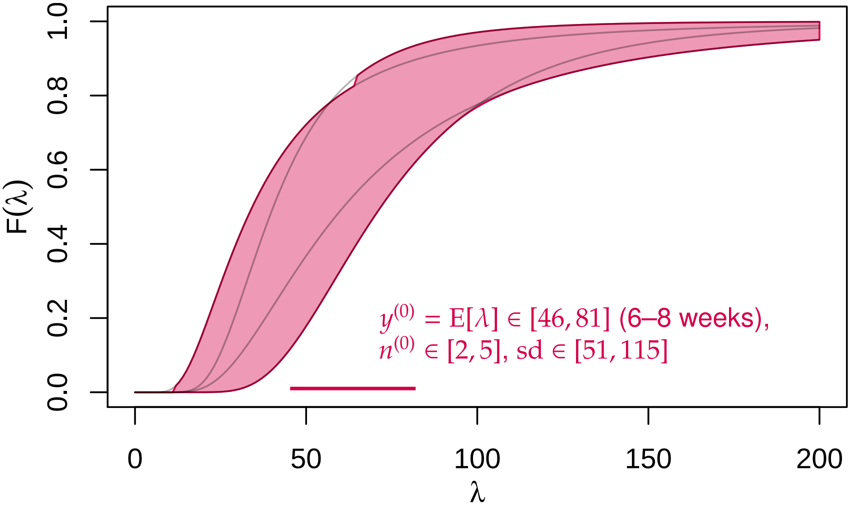
Walter and Augustin (2009), Walter (2013):
$\Pi^{(0)} = [\underline{n}^{(0)}, \overline{n}^{(0)}] \times [\underline{y}^{(0)}, \overline{y}^{(0)}]$
gives tractability & meaningful reaction to prior-data conflict:

- larger set of posteriors
- more imprecise / cautious probability statements

Sets of prior distributions: examples



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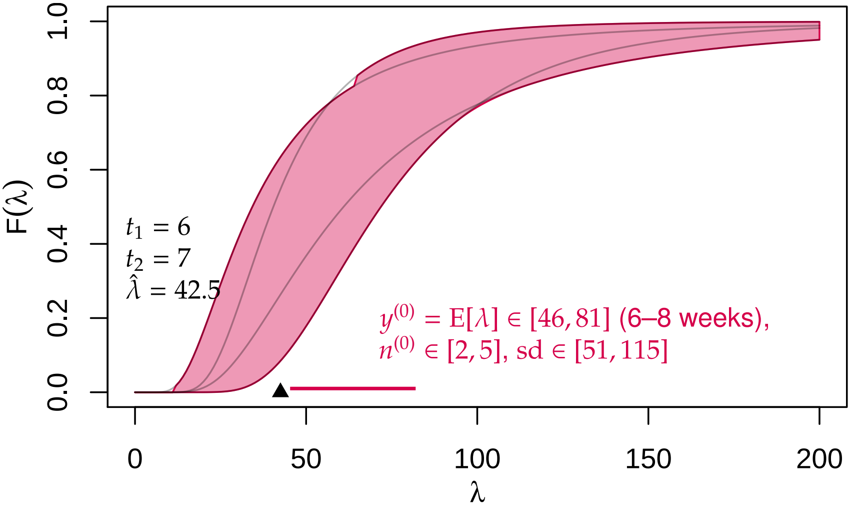

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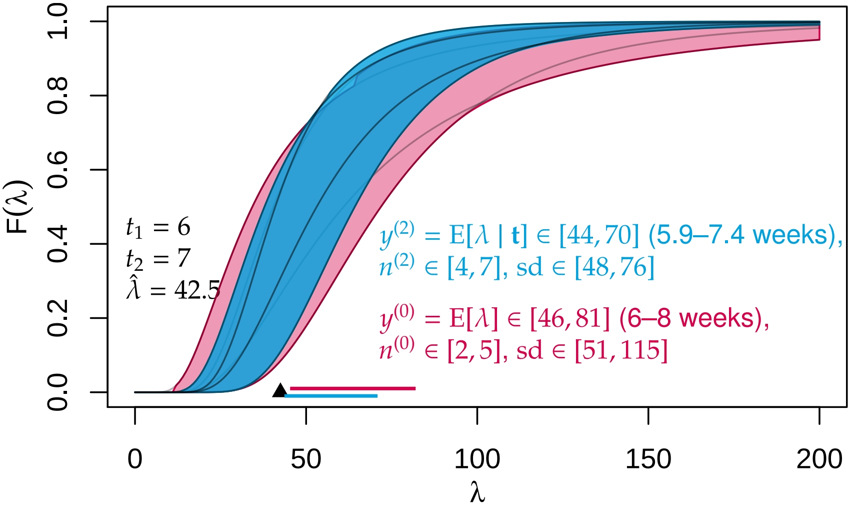

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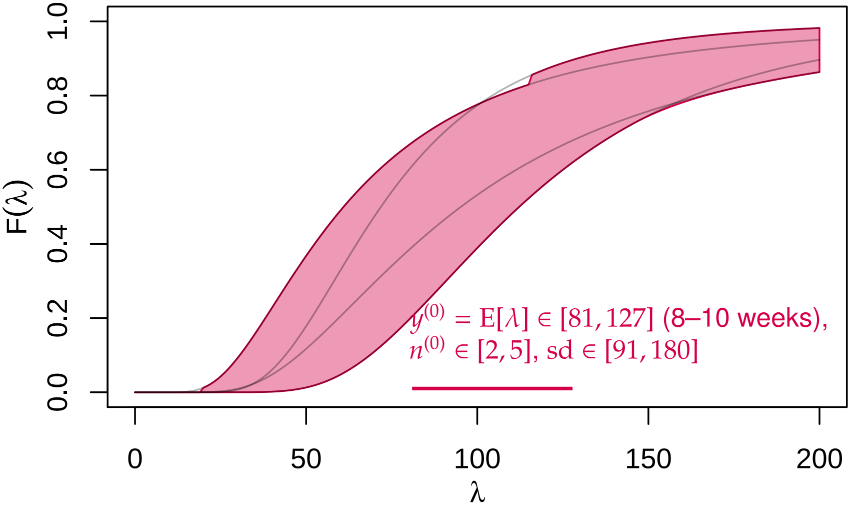

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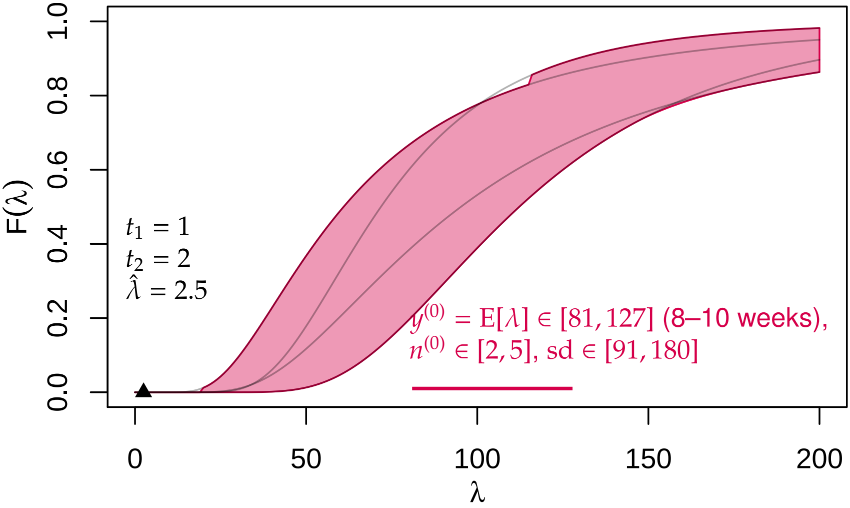

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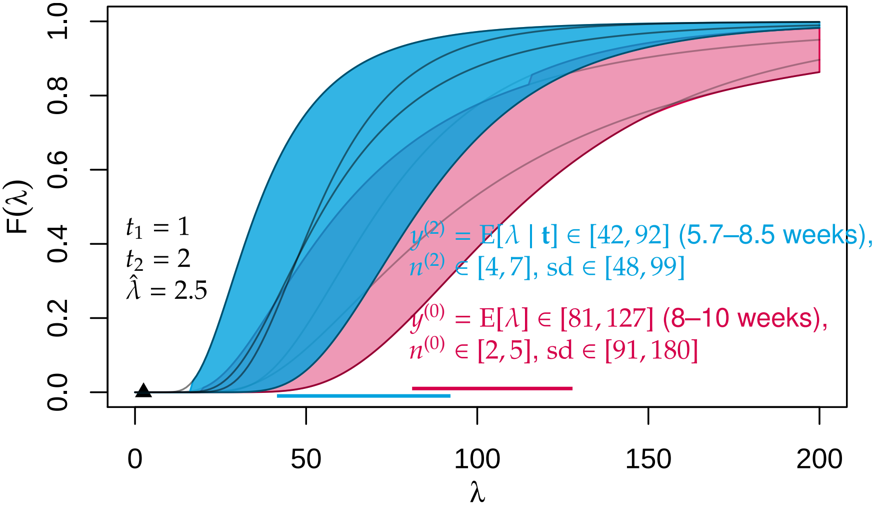

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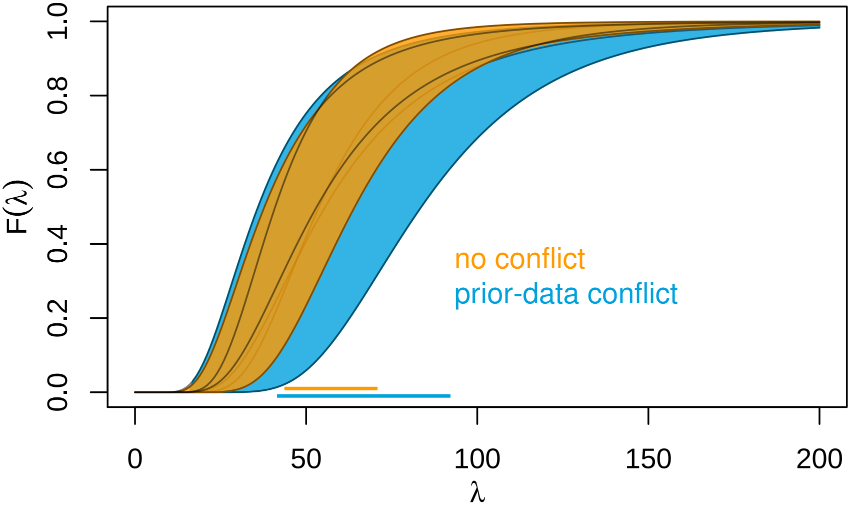

System reliability

- Closed form for the system reliability:

$$R_{\mathsf{sys}}(t \mid \boldsymbol{t}_m^{\ell}, n^{(0)}, y^{(0)})$$

$$= 1 - \sum_{i=0}^{\ell-m} (-1)^i \binom{\ell-m}{i} \left(\frac{n^{(0)} y^{(0)} + (\ell-m) t_{\mathsf{now}}^k + \sum_{j=1}^m t_j^k}{n^{(0)} y^{(0)} + (\ell-m-i) t_{\mathsf{now}}^k + \sum_{j=1}^m t_j^k + i t^k} \right)^{n^{(0)}+m+1}$$

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- Lower / upper bound through optimization for each t:

$$\underline{R}_{\mathsf{sys}}(t \mid \boldsymbol{t}^{\ell}_{m}, \Pi^{(0)}) = \min_{n^{(0)} \in [\underline{n}^{(0)}, \overline{n}^{(0)}]} R_{\mathsf{sys}}(t \mid \boldsymbol{t}^{\ell}_{m}, n^{(0)}, \underline{y}^{(0)})$$

$$\overline{R}_{\mathsf{sys}}(t \mid \boldsymbol{t}^{\ell}_{m}, \Pi^{(0)}) = \max_{n^{(0)} \in [\underline{n}^{(0)}, \overline{n}^{(0)}]} R_{\mathsf{sys}}(t \mid \boldsymbol{t}^{\ell}_{m}, n^{(0)}, \overline{y}^{(0)})$$

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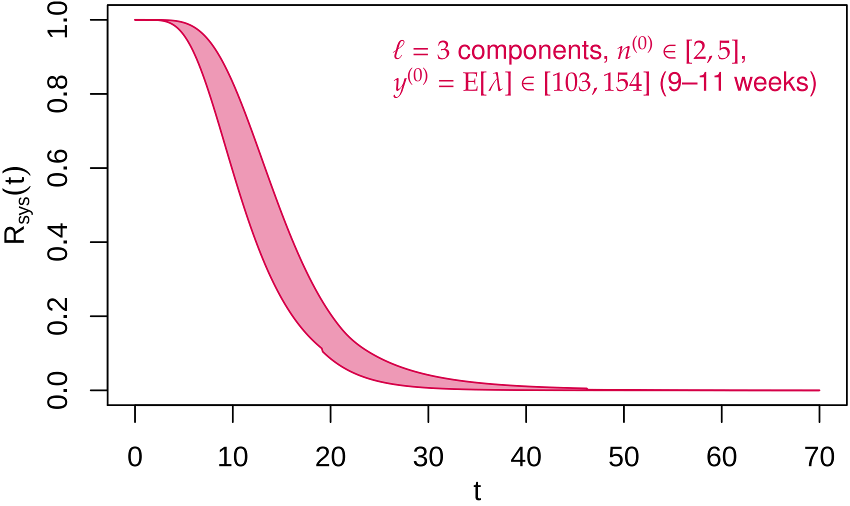

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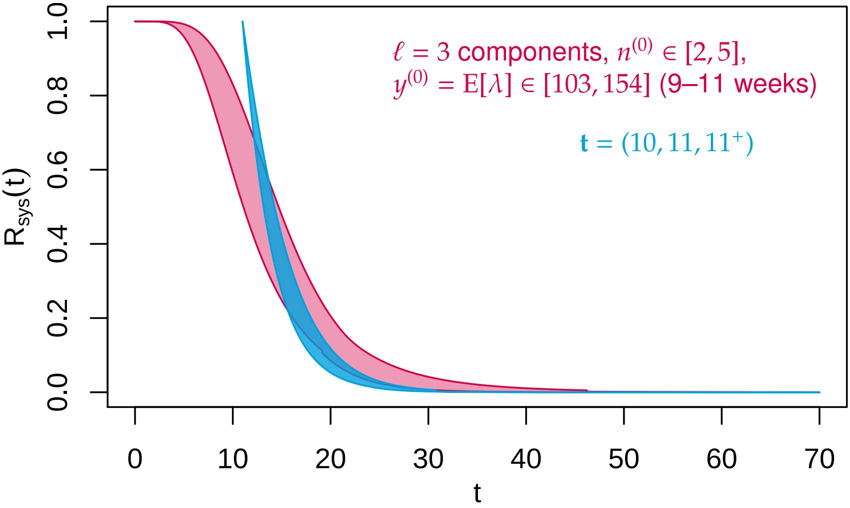

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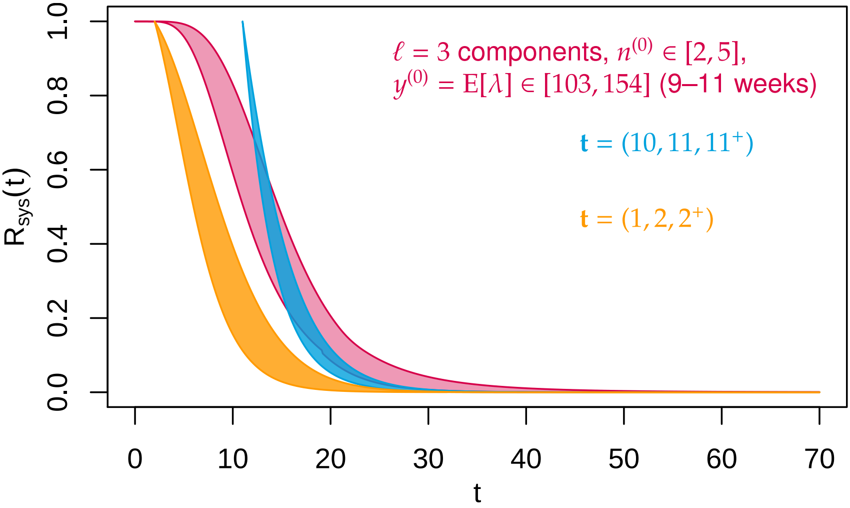

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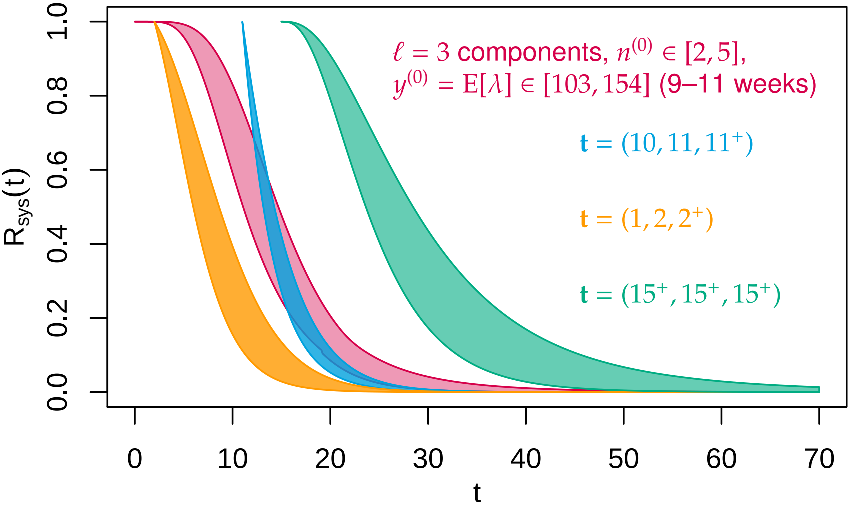

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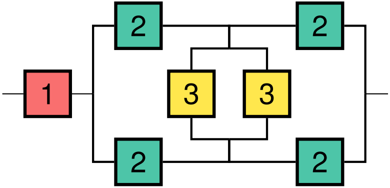

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- Allow dependence between components (common-cause failure, . . .)
- Use model for maintenance planning

References

Coolen, Frank P. A. and Tahani Coolen-Maturi (2012). "Generalizing the Signature to Systems with Multiple Types of Components". In: *Complex Systems and Dependability*. Ed. by W. Zamojski et al. Vol. 170. Advances in Intelligent and Soft Computing. Springer, pp. 115–130. DOI: 10.1007/978-3-642-30662-4_8.

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General systems via the survival signature

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System Reliability Estimation under Prior-Data Conflict

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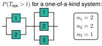
[b] Department of Mathematical Sciences, Durham University, Durham, UK

System reliability

We want to find the *system reliability* $P(T_{\text{sys}} > t)$ for a one-of-a-kind system:



The system consists of n_k exchangeable components of types 1, …, K.

Component Lifetimes

The lifetime for each k is assumed as Weibull with fixed shape β:

$$F_k(t \mid \lambda_k) = 1 - e^{-\frac{t^\beta}{\lambda_k}}$$
$$\mathrm{E}[T \mid \lambda_k] = \sqrt[\beta]{\lambda_k}\,\Gamma(1 + 1/\beta)$$

We have information on λ_k from the component manufacturer, but do not fully trust it and model knowledge on λ_k cautiously with a *set of priors* $\mathcal{M}_k^{(0)}$.

Set of Priors

Each $\mathcal{M}_k^{(0)}$ is taken as a set of conjugate inverse Gamma priors. In terms of canonical parameters $n^{(0)}, y^{(0)}$, $\mathcal{M}_k^{(0)} = \{ \mathrm{IG}(n_k^{(0)} + 1, n_k^{(0)} y_k^{(0)}) \mid [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}] \}$, where $y_k^{(0)} = \mathrm{E}[\lambda_k \mid n_k^{(0)}, y_k^{(0)}]$ and $n_k^{(0)}$ = pseudocounts. The prior parameter set $\Pi_k^{(0)} = [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}]$ allows for more imprecision in case of *prior-data conflict* [2].

Data

We observe the system from startup until t_{now}. For each k, the data $\mathbf{t}^k_{e_k;n_k}$ consists of e_k failure times and $n_k - e_k$ censored observations. $n_k^{(0)}$ and $y_k^{(0)}$ are updated to $n_k^{(n)}$ and $y_k^{(n)}$ via Bayes' Rule.

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$$\underline{P}\left(T_{\text{sys}} > t \mid \{n_k^{(0)}, y_k^{(0)}, \mathbf{t}^k_{e_k;n_k}\}^{1:K}\right) = \min_{n_1^{(0)},\dots,n_K^{(0)}} \sum_{l_1=0}^{n_1-e_1} \cdots \sum_{l_K=0}^{n_K-e_K} \Phi(l_1,\dots,l_K) \prod_{k=1}^{K} P(C_t^k = l_k \mid n_k^{(0)}, \underline{y}_k^{(0)}, \mathbf{t}^k_{e_k;n_k})$$

Survival signature $\Phi(l_1, \dots, l_K)$ [1]
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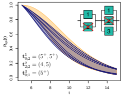
l_1	l_2	l_3	Φ	l_1	l_2	l_3	Φ
0	0	0	0	0	2	1	1
0	0	1	0	1	0	0	0
0	1	0	0	1	0	1	0.5
0	1	1	0.5	1	1	1	0.75
0	2	0	1	⋮	⋮	⋮	⋮

Posterior predictive probability that l_k of the $n_k - e_k$ surviving k's function at time t:

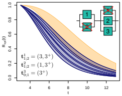
$$\binom{n_k - e_k}{l_k} \int \left[P_k(T > t \mid T > t_{\text{now}}, \lambda_k)\right]^{l_k} \times \left[1 - P_k(T > t \mid T > t_{\text{now}}, \lambda_k)\right]^{n_k - e_k - l_k} f_{\lambda_k \mid \dots}(\lambda_k \mid n_k^{(0)}, y_k^{(0)}, \mathbf{t}^k_{e_k;n_k})\, \mathrm{d}\lambda_k$$
$$= \binom{n_k - e_k}{l_k} \sum_{j=0}^{n_k - e_k - l_k} (-1)^j \binom{n_k - e_k - l_k}{j} \left(\frac{n_k^{(n)} y_k^{(n)}}{n_k^{(n)} y_k^{(n)} + (l_k + j)(t^\beta - (t_{\text{now}})^\beta)} \right)^{n_k^{(n)} + 1}$$

We assume $\beta = 2$, $\mathrm{E}[T \mid y_1^{(0)}] \in [9, 11]$, $n_1^{(0)} \in [2, 10]$, $\mathrm{E}[T \mid y_2^{(0)}] \in [4, 5]$, $n_2^{(0)} \in [8, 16]$, and $\mathrm{E}[T \mid y_3^{(0)}] \in [9, 11]$, $n_3^{(0)} \in [1, 5]$.

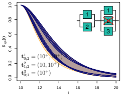
Failure times as expected



Surprisingly early failures



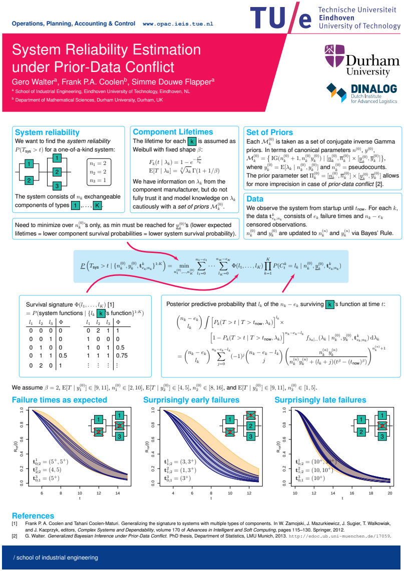
Surprisingly late failures



References

[1] Frank P. A. Coolen and Tahani Coolen-Maturi. Generalizing the signature to systems with multiple types of components. In W. Zamojski, J. Mazurkiewicz, J. Sugier, T. Walkowiak, and J. Kacprzyk, editors, *Complex Systems and Dependability*, volume 170 of *Advances in Intelligent and Soft Computing*, pages 115–130. Springer, 2012.
[2] G. Walter. *Generalized Bayesian Inference under Prior-Data Conflict*. PhD thesis, Department of Statistics, LMU Munich, 2013. http://edoc.ub.uni-muenchen.de/17059.

General systems via the survival signature



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Technische Universiteit Eindhoven University of Technology

System Reliability Estimation under Prior-Data Conflict

Gero Walter[a], Frank P.A. Coolen[b], Simme Douwe Flapper[a]

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[b] Department of Mathematical Sciences, Durham University, Durham, UK

Durham University

DINALOG Dutch Institute for Advanced Logistics

System reliability

We want to find the *system reliability* $P(T_{\text{sys}} > t)$ for a one-of-a-kind system:

$n_1 = 2$, $n_2 = 2$, $n_3 = 1$

The system consists of n_k exchangeable components of types 1, …, K.

Component Lifetimes

The lifetime for each k is assumed as Weibull with fixed shape β:

$$F_k(t \mid \lambda_k) = 1 - e^{-\frac{t^\beta}{\lambda_k}}$$
$$\mathrm{E}[T \mid \lambda_k] = \sqrt[\beta]{\lambda_k}\,\Gamma(1 + 1/\beta)$$

We have information on λ_k from the component manufacturer, but do not fully trust it and model knowledge on λ_k cautiously with a *set of priors* $\mathcal{M}_k^{(0)}$.

Set of Priors

Each $\mathcal{M}_k^{(0)}$ is taken as a set of conjugate inverse Gamma priors. In terms of canonical parameters $n^{(0)}$, $y^{(0)}$,
$\mathcal{M}_k^{(0)} = \{ \mathrm{IG}(n_k^{(0)} + 1, n_k^{(0)} y_k^{(0)}) \mid [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}] \}$,
where $y_k^{(0)} = \mathrm{E}[\lambda_k \mid n_k^{(0)}, y_k^{(0)}]$ and $n_k^{(0)}$ = pseudocounts.
The prior parameter set $\Pi_k^{(0)} = [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}]$ allows for more imprecision in case of *prior-data conflict* [2].

Data

We observe the system from startup until t_{now}. For each k, the data $\mathbf{t}^k_{e_k;n_k}$ consists of e_k failure times and $n_k - e_k$ censored observations.
$n_k^{(0)}$ and $y_k^{(0)}$ are updated to $n_k^{(n)}$ and $y_k^{(n)}$ via Bayes' Rule.

Need to minimize over $n_k^{(0)}$'s only, as min must be reached for $\underline{y}_k^{(0)}$'s (lower expected lifetimes = lower component survival probabilities = lower system survival probability).

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Survival signature $\Phi(l_1,\dots,l_K)$ [1]
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Failure times as expected

$\mathbf{t}^3_{0;1} = (5^+)$

Surprisingly early failures

$\mathbf{t}^1_{1;2} = (3, 3^+)$, $\mathbf{t}^2_{1;2} = (1, 3^+)$, $\mathbf{t}^3_{0;1} = (3^+)$

Surprisingly late failures

$\mathbf{t}^3_{0;1} = (10^+)$

References

[1] Frank P. A. Coolen and Tahani Coolen-Maturi. Generalizing the signature to systems with multiple types of components. In W. Zamojski, J. Mazurkiewicz, J. Sugier, T. Walkowiak, and J. Kacprzyk, editors, *Complex Systems and Dependability*, volume 170 of *Advances in Intelligent and Soft Computing*, pages 115–130. Springer, 2012.

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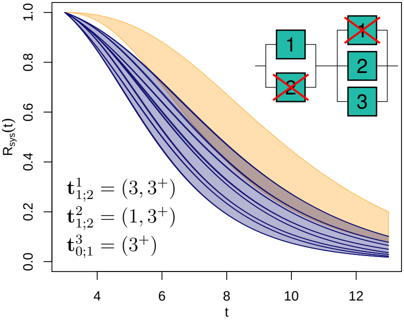
/ school of industrial engineering

Surprisingly early failures



General systems via the survival signature

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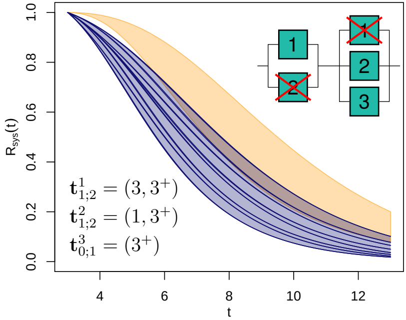

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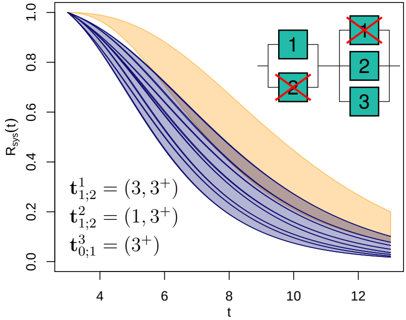

$$\underline{P}\left(T_{\text{sys}} > t \mid \{n_k^{(0)}, y_k^{(0)}, \mathbf{t}^k_{e_k;n_k}\}^{1:K}\right)$$

$$= \min_{n_1^{(0)},\ldots,n_K^{(0)}} \sum_{l_1=0}^{n_1-e_1} \cdots \sum_{l_K=0}^{n_K-e_K} \Phi(l_1,\ldots,l_K) \prod_{k=1}^{K} P(C_t^k = l_k \mid n_k^{(0)}, \underline{y}_k^{(0)}, \mathbf{t}^k_{e_k;n_k})$$

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