"Strong happiness" and other properties of certain imprecise probability models when treating samples sequentially

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Prior-Data Conflict

Prior-Data Conflict $\widehat{=}$ situation in which. . .

- . . . *informative prior beliefs* and *trusted data* (sampling model correct, no outliers, etc.) are in conflict
- ". . . the prior [places] its mass primarily on distributions in the sampling model for which the observed data is surprising." (Evans & Moshonov, 2006)

Simple Example: Dirichlet-Multinomial-Model

Data:	$\mathbf{k}$	$\sim$	$\mathrm{M}(\boldsymbol{\theta})$	$(\sum k_j = n)$
conjugate prior:	$\boldsymbol{\theta}$	$\sim$	$\mathrm{Dir}(\boldsymbol{\alpha})$	$(\sum \theta_j = 1)$
posterior:	$\boldsymbol{\theta} \mid \mathbf{k}$	$\sim$	$\mathrm{Dir}(\boldsymbol{\alpha} + \mathbf{k})$	

$$\mathbb{E}[\theta_j] = \frac{\alpha_j}{\sum \alpha_i} \qquad \mathbb{V}(\theta_j) = \frac{\mathbb{E}[\theta_j](1 - \mathbb{E}[\theta_j])}{\sum \alpha_i + 1}$$

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$$y_j^{(1)} = \frac{n^{(0)}}{n^{(0)} + n} \cdot y_j^{(0)} + \frac{n}{n^{(0)} + n} \cdot \frac{k_j}{n} \qquad n^{(1)} = n^{(0)} + n$$

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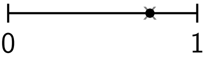
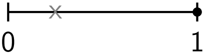
Case (i): $y_j^{(0)} = 0.75\,, \quad (n^{(0)} = 8)$ $\qquad k_j/n = 0.75 \quad (n = 16)$



Case (ii): $y_j^{(0)} = 0.25\,, \quad (n^{(0)} = 8)$ $\qquad k_j/n = 1 \quad (n = 16)$



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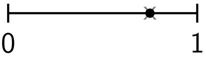

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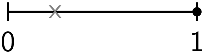
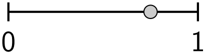


$\rightarrow \quad \mathbb{E}[\theta_j \mid \mathbf{k}] = y_j^{(1)} = 0.75, \quad \mathbb{V}(\theta_j \mid \mathbf{k}) = 3/400$



$(\mathbb{V}(\theta_j) = 1/48)$

⚠ Posterior inferences do not reflect uncertainty due to unexpected observations! ⚠

Conjugate Priors

Weighted average structure is underneath *all common* conjugate priors for exponential family sampling distributions!

$X \overset{iid}{\sim}$ linear, canonical exponential family, i.e.

$$p(x \mid \theta) \propto \exp\left\{\langle\psi, \tau(x)\rangle - n\mathbf{b}(\psi)\right\} \qquad \left[\psi \text{ transformation of } \theta\right]$$

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→ (conjugate) posterior:

$$p(\theta \mid x) \propto \exp\left\{n^{(1)}\left[\langle\psi, y^{(1)}\rangle - \mathbf{b}(\psi)\right]\right\},$$

where $y^{(1)} = \frac{n^{(0)}}{n^{(0)} + n} \cdot y^{(0)} + \frac{n}{n^{(0)} + n} \cdot \frac{1}{n}\tau(x)$ and $n^{(1)} = n^{(0)} + n$.

Generalized iLUCK-models

Model for Bayesian inference with sets of priors
(Walter & Augustin, 2009)

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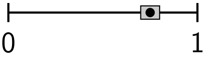
2. construct sets of priors via sets of parameters $y^{(0)} \in \mathcal{Y}^{(0)} \times n^{(0)} \in \mathcal{N}^{(0)}$
3. set of posteriors $\hat{=}$ set of (element-wise) updated priors
   ➥ still easy to handle: described as set of $(y^{(1)}, n^{(1)})$'s

$$y^{(1)} = \frac{n^{(0)}}{n^{(0)} + n} \cdot y^{(0)} + \frac{n}{n^{(0)} + n} \cdot \frac{1}{n}\tau(x)$$

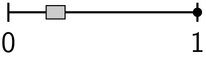
$$n^{(1)} = n^{(0)} + n$$

Generalized iLUCK-models: Simple Example

Case (i): $y_j^{(0)} \in [0.7, 0.8]$, $(n^{(0)} \in [1, 8])$ $k_j/n = 0.75$ $(n = 16)$



Case (ii): $y_j^{(0)} \in [0.2, 0.3]$, $(n^{(0)} \in [1, 8])$ $k_j/n = 1$ $(n = 16)$



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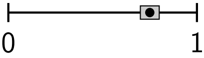
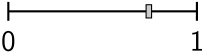


→ $y_j^{(1)} \in [0.73, 0.76]$ $(n^{(0)} \in [17, 24])$



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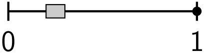

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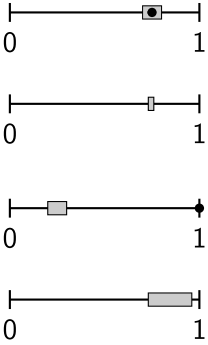

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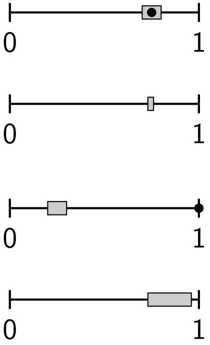

Generalized iLUCK-models lead to cautious inferences if, and only if, caution is needed.

Generalized iLUCK-models: Parameter Sets

Prior parameter set $y^{(0)} \in \mathcal{Y}^{(0)} \times n^{(0)} \in \mathcal{N}^{(0)}$ is rectangular, but posterior parameter set

$$\left\{ (y^{(1)}, n^{(1)}) \,\middle|\, y^{(1)} = \frac{n^{(0)} y^{(0)} + \tau(x)}{n^{(0)} + n},\; n^{(1)} = n^{(0)} + n, \right.$$
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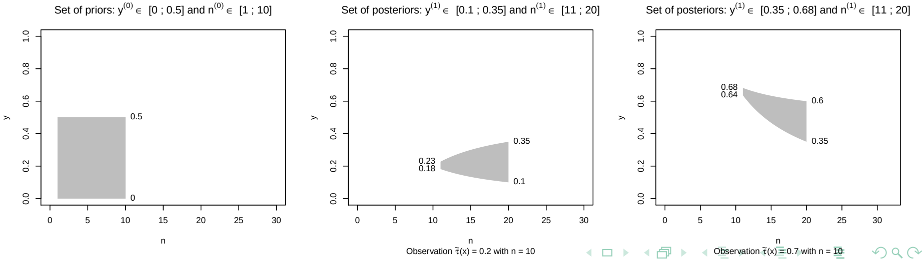
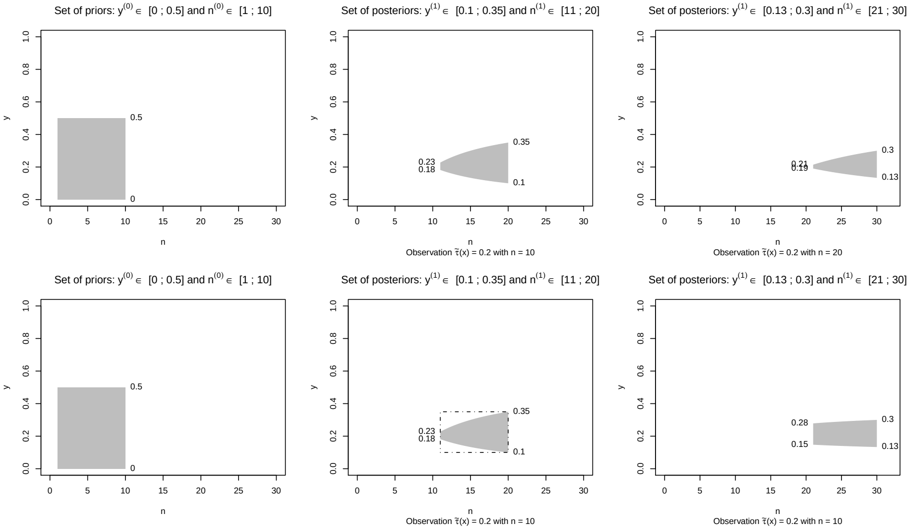


Generalized iLUCK-models: Direct vs. Sequential



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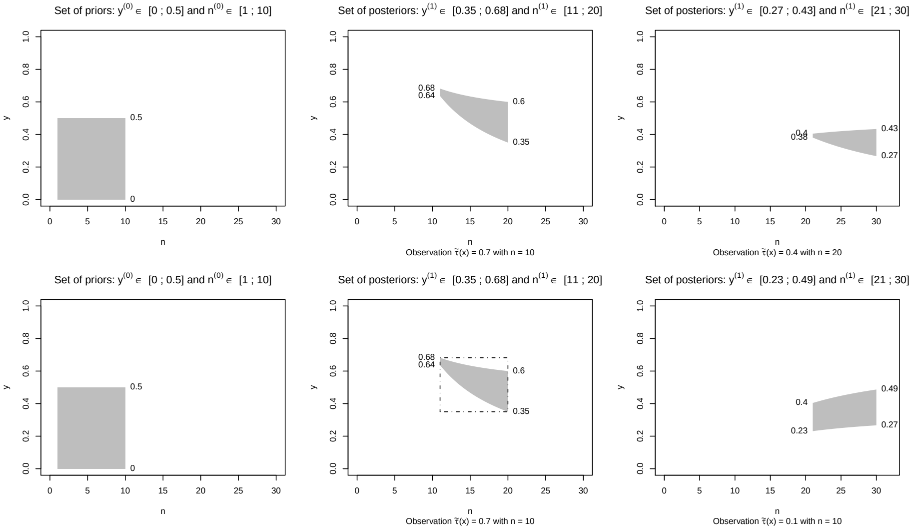

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- more information (order) → more imprecise posterior!

In general: $\mathcal{Y}^{(2)}_{\text{dir}} \subseteq \mathcal{Y}^{(2)}_{\text{seq}}$

if no pdc in both steps:	$\mathcal{Y}^{(2)}_{\text{dir}} = \mathcal{Y}^{(2)}_{\text{seq}}$
if pdc in both steps in same direction:	$\mathcal{Y}^{(2)}_{\text{dir}} = \mathcal{Y}^{(2)}_{\text{seq}}$
if neither of the former:	$\mathcal{Y}^{(2)}_{\text{dir}} \subset \mathcal{Y}^{(2)}_{\text{seq}}$

especially: if prior-data conflict 'on the way', then $\mathcal{Y}^{(2)}_{\text{dir}} \subset \mathcal{Y}^{(2)}_{\text{seq}}$!

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(but: no need to use $\mathcal{Y}^{(2)}_{\text{seq}}$ for inference if you think, e.g, that the sequence of data is exchangeable nevertheless.)

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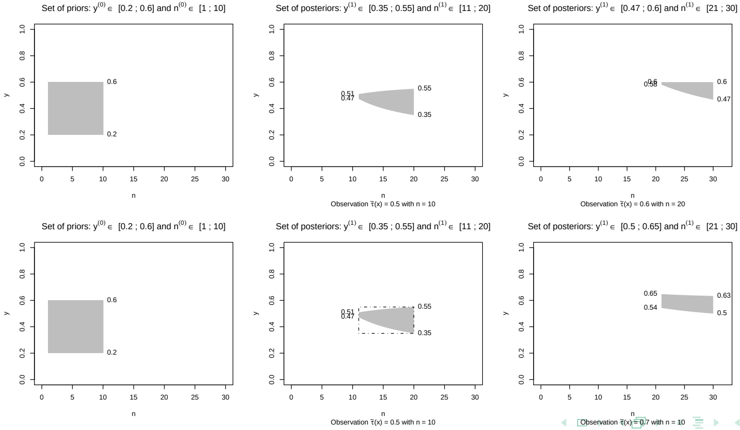

Generalized iLUCK-models: Interval Lengths for 3 Steps

Data situation: Bernoulli sampling (observe 0 or 1)
Start with $\mathcal{Y}^{(0)} = [0, 1]$

Compare sequences:

- A $(1, 0, 1)$
- B $(1, 1, 0)$
- C $(0, 1, 1)$
- D $([2/3, n = 3])$ (direct)

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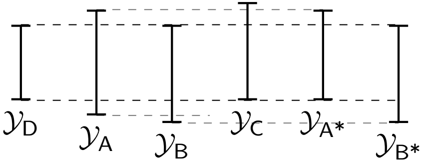

Interval length for y: "$|\mathcal{Y}|$" $=: I$

$I_A > I_B$, $I_A > I_C$

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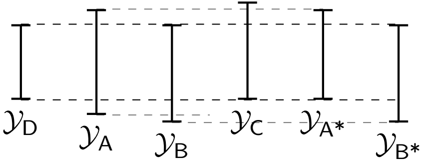

Interval length for y: "$|\mathcal{Y}|$" $=: l$

$$l_A > l_B,\ l_A > l_C$$

$A^{**} = B^{**}$ $(1, [1/2, n = 2]) \hat{=} D$ (if $\overline{n}^{(0)} \geq 1$)

C^{**} $(0, [1, n = 2]) \hat{=} C$

"Strong Happiness"

Data situation: Bernoulli sampling (observe 0 or 1).

Idea: Choose sample size n_1 such that $I^{(1)}$ low enough under certain threshold I such that – even if we got another sample n_2 in conflict to $\mathcal{Y}^{(1)}$ – $I^{(2)}$ is still below I!

With a sample of such a size n_1 we attain "strong happiness", because whatever we would see as a following sample, we would never get more imprecise than I!

(Is only possible because degree of prior-data conflict is bounded due to Bernoulli sampling!).

"Strong Happiness"

$$I^{(2)} = \frac{\overline{n}^{(0)} I^{(0)}}{\overline{n}^{(0)} + n_1 + n_2} + \frac{\overline{n}^{(0)} - \underline{n}^{(0)}}{\overline{n}^{(0)} + n_1 + n_2} \left(\frac{n_1}{\underline{n}^{(0)} + n_1} \Delta\left(\frac{c_1}{n_1}, \mathcal{Y}^{(0)}\right) + \frac{n_2}{\underline{n}^{(0)} + n_1 + n_2} \Delta\left(\frac{c_2}{n_2}, \mathcal{Y}^{(1)}\right) \right)$$

$$\overset{!}{\leq} I \quad \forall \, (k_2, \, n_2)$$

"Strong Happiness"

$$I^{(2)} = \frac{\overline{n}^{(0)} I^{(0)}}{\overline{n}^{(0)} + n_1 + n_2} + \frac{\overline{n}^{(0)} - \underline{n}^{(0)}}{\overline{n}^{(0)} + n_1 + n_2} \left(\frac{n_1}{\underline{n}^{(0)} + n_1} \Delta\left(\frac{c_1}{n_1}, \mathcal{Y}^{(0)}\right) + \frac{n_2}{\underline{n}^{(0)} + n_1 + n_2} \Delta\left(\frac{c_2}{n_2}, \mathcal{Y}^{(1)}\right) \right)$$

$$\overset{!}{\leq} I \quad \forall\, (k_2,\, n_2)$$

If data from second sample in conflict with $\mathcal{Y}^{(1)}$ (and conflict strong enough), then imprecision $I^{(2)}$ should first increase and then decrease in n_2.

➡ find n_2 that maximizes $I^{(2)}$ for maximal possible conflict with current $\mathcal{Y}^{(1)}$, plug into formula for $I^{(2)}$ and give n_1 as a function of $I^{(0)}$ (or $\mathcal{Y}^{(0)}$) and I (and $\mathcal{N}^{(0)}$).

Plans, Further Ideas

- Try to attain Strong Happiness by sorting out formula
- Write programs for Strong Happiness
- Investigate interval lengths in general situation (not only Bernoulli sampling)
- Focus considerations on certain event of interest instead of $\mathcal{Y}$?
- ...