On Prior-Data Conflict in Predictive Bernoulli Inferences

Gero Walter, Thomas Augustin

Department of Statistics
Ludwig-Maximilians-Universität München (LMU)

Frank P.A. Coolen

Department of Mathematics
Durham University

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- additional to observations, we have strong prior information (we are convinced that P(heads) should be around 0.75)
- interested in probability P that the next observation is a head. (predictive probability!)
- **prior-data conflict:** if P(heads) for the coin is actually very different from our prior guess (i.e., prior information and data are in conflict), this should show up in the predictive inferences (probability P and, e.g., confidence intervals)

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conjugate prior:	p	$\sim$	$\text{Beta}(n^{(0)}, y^{(0)})$
posterior:	$p \mid s$	$\sim$	$\text{Beta}(n^{(n)}, y^{(n)})$

$$y^{(n)} = \frac{n^{(0)}}{n^{(0)} + n} \cdot y^{(0)} + \frac{n}{n^{(0)} + n} \cdot \frac{s}{n}, \qquad n^{(n)} = n^{(0)} + n$$

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$$\text{Var}(p \mid s) = \frac{y^{(n)}(1 - y^{(n)})}{n^{(n)} + 1}$$ → no reaction to prior-data conflict!

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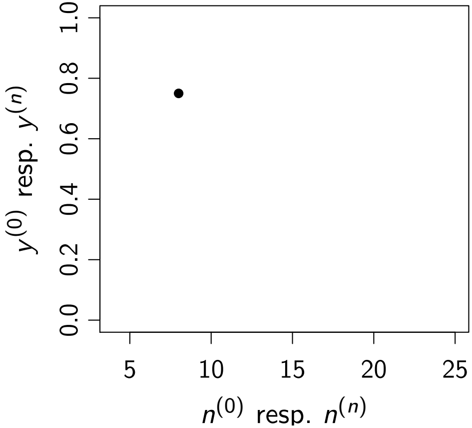

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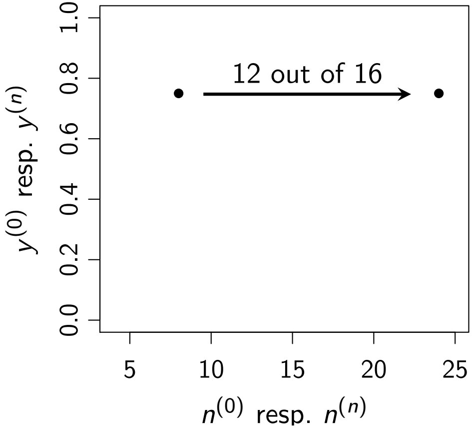

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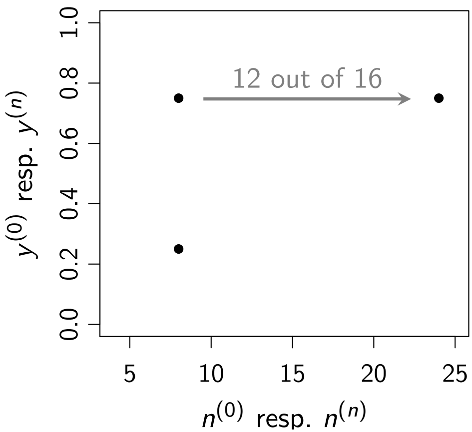

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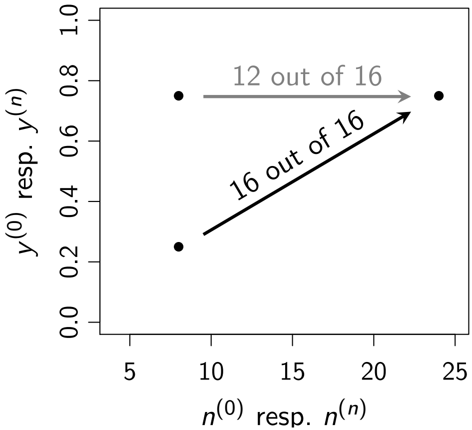

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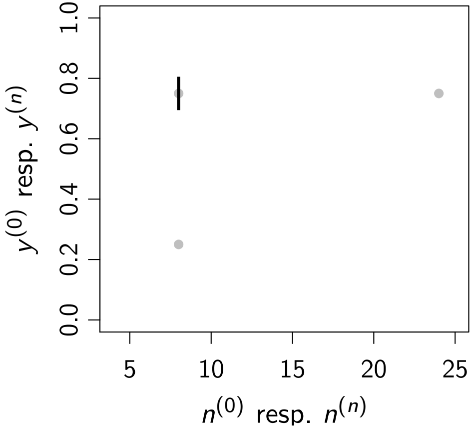

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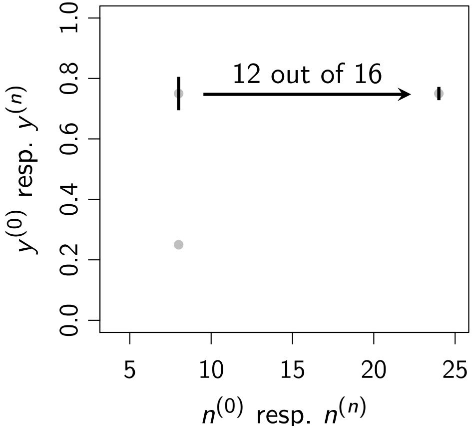

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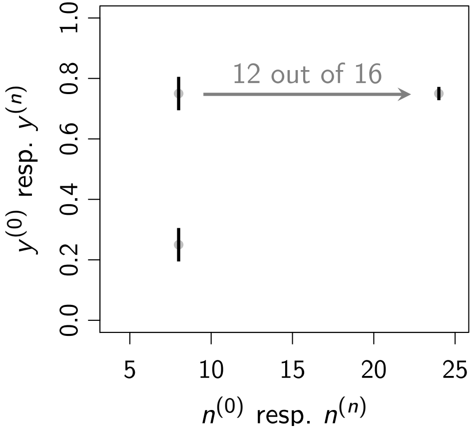

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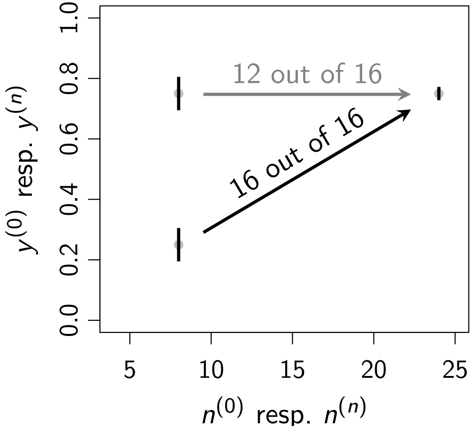

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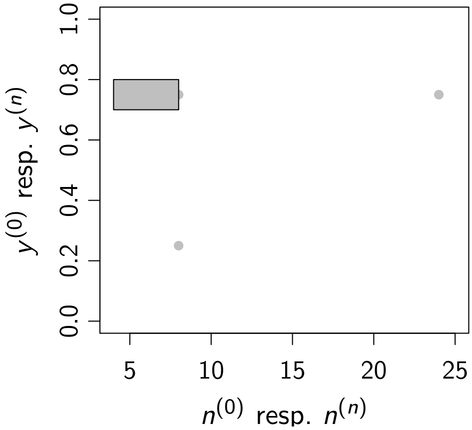

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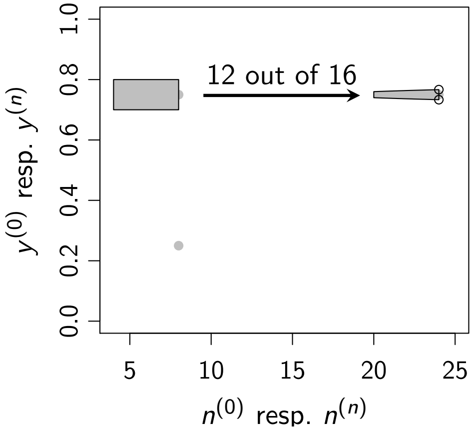

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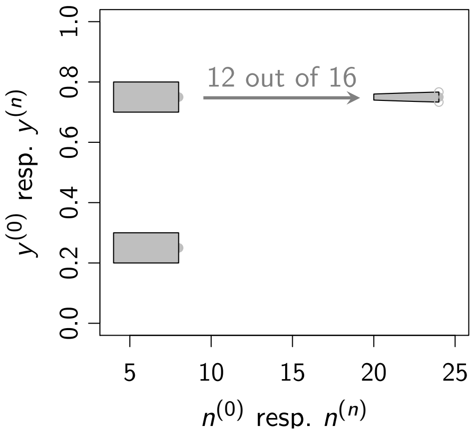

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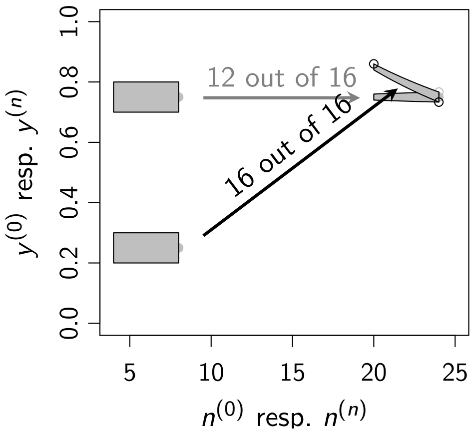

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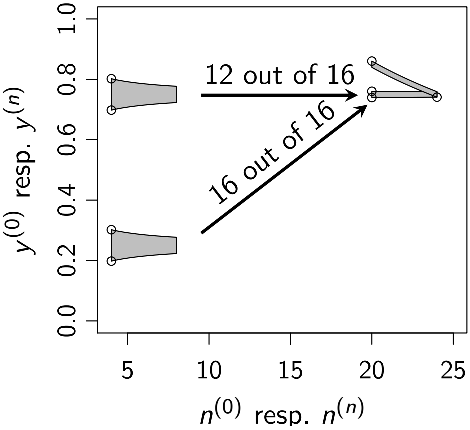

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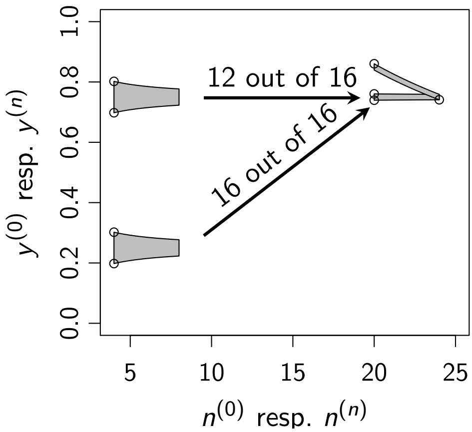
  ➥ the "anteater" shape:
- "anteater" is difficult to elicit
  ➥ other approach: combine predictive inferences



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Combine predictive inferences of

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by weighing them **imprecisely**:

$$\underline{\mathrm{P}} = \min_{\alpha \in [\alpha_l, \alpha_r]} \underline{\mathrm{P}}_\alpha, \qquad \text{where} \qquad \underline{\mathrm{P}}_\alpha = \alpha \underline{\mathrm{P}}^i + (1-\alpha)\underline{\mathrm{P}}^u$$

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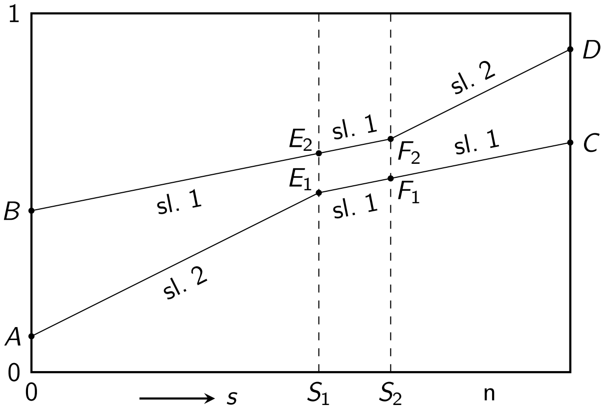
Comparison with IBBMs



The predictive probability plot (PPP) illustrates model behaviour.

Summary, Outlook

IBBMs

- sets of BBMs: vary parameters $n^{(0)}$, $y^{(0)}$
- can be generalized to any distribution from exponential family: same weighted average structure for $y^{(n)}$ (Quaghebeur & de Cooman 2005)
- generalization of pdc-IBBM: Walter, Augustin (2009)

Weighted Inference

- combine BBM inferences, not BBM models
- can be used to combine any two predictive inferences from any model, on any event of interest
- possible source for P^u / P^i: NPI (Coolen & Augustin 2009)