Sets of canonical parameters in imprecise Bayesian inference with conjugate priors

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November 1st, 2011

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- additional to observations, we have strong prior information (we are convinced that P(heads) should be around 0.75)
- interested in certain inferences, e.g. probability P that the next observation is a head.
- **prior-data conflict:** if P(heads) for the coin is actually very different from our prior guess (i.e., prior information and data are in conflict), this should show up in the inferences (probability P and, e.g., confidence intervals)

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conjugate prior:	p	$\sim$	$\text{Beta}(n^{(0)}, y^{(0)})$
posterior:	$p \mid s$	$\sim$	$\text{Beta}(n^{(n)}, y^{(n)})$

$$y^{(n)} = \frac{n^{(0)}}{n^{(0)} + n} \cdot y^{(0)} + \frac{n}{n^{(0)} + n} \cdot \frac{s}{n}\,, \qquad n^{(n)} = n^{(0)} + n$$

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$$\text{Var}(p \mid s) = \frac{y^{(n)}(1 - y^{(n)})}{n^{(n)} + 1}$$

➡ no reaction to prior-data conflict!

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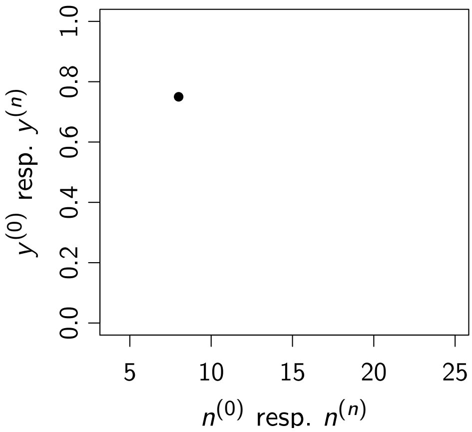

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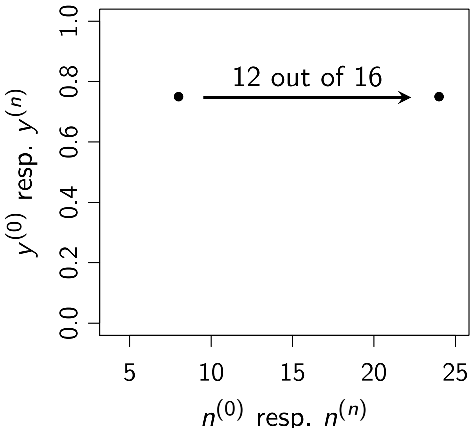

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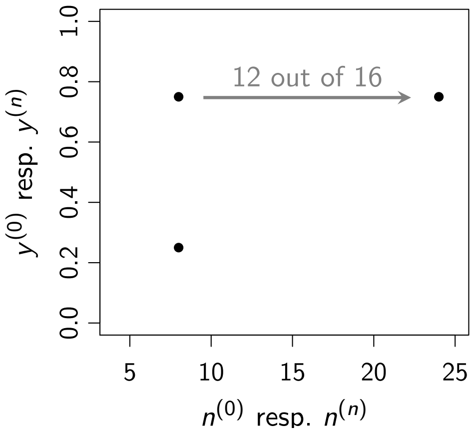

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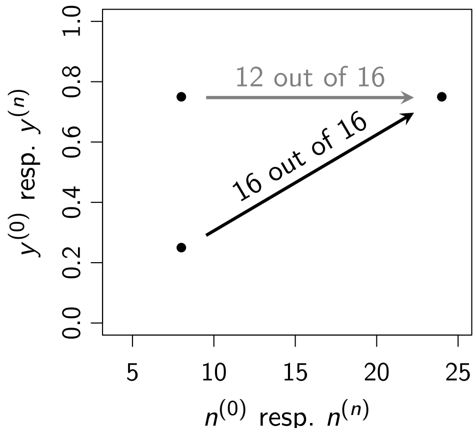

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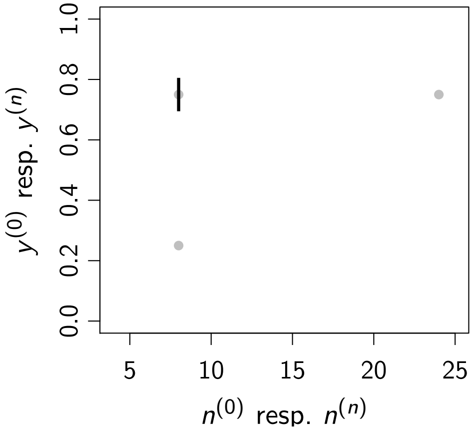

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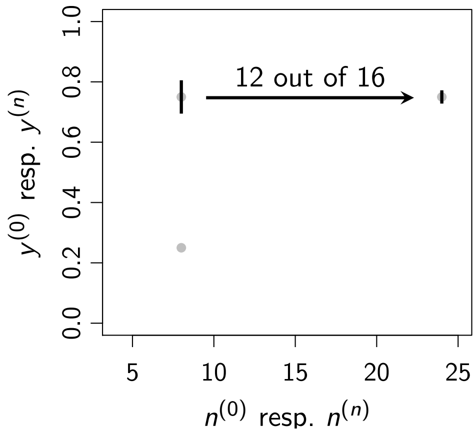

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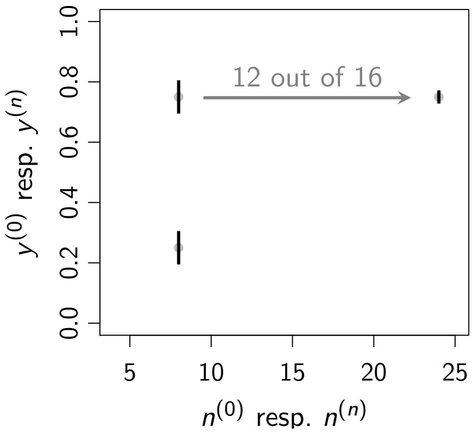

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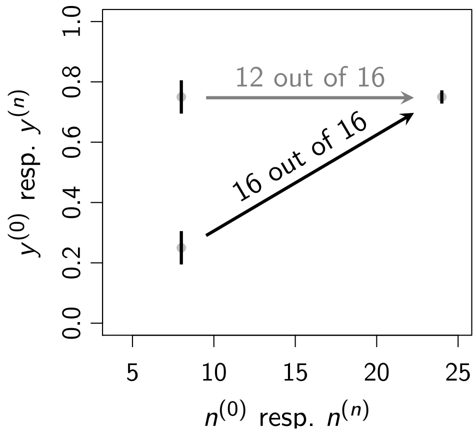

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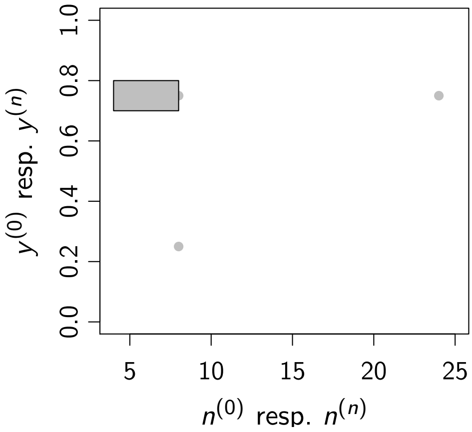

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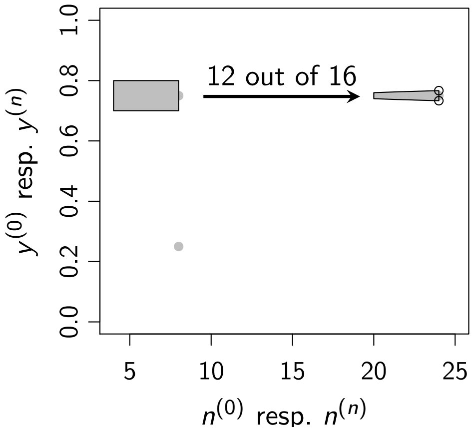

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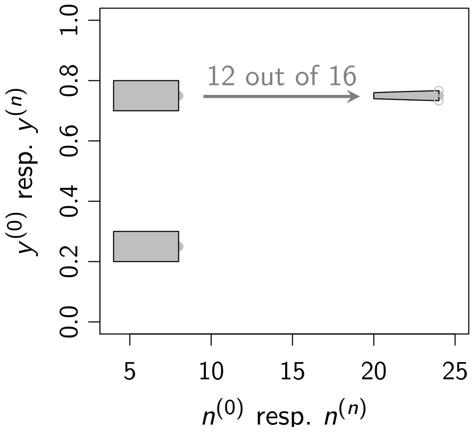

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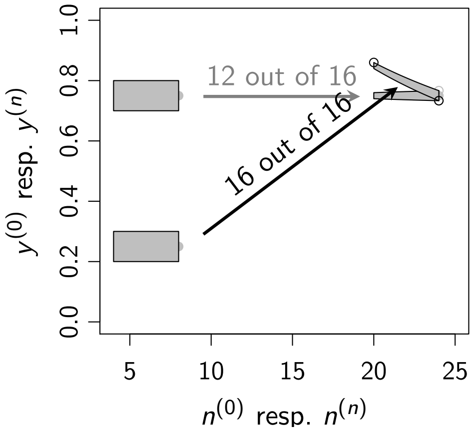

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- open ends:
  - why rectangles? why same $y^{(0)}$ interval at $\underline{n}^{(0)}$ and $\overline{n}^{(0)}$?
  - 'conjugate' set description possible? (invariant under updating)
  - what could be an easy elicitation procedure for parameter sets?

Allow different $y^{(0)}$ intervals at $\underline{n}^{(0)}$ and $\overline{n}^{(0)}$:

"snout left" (shorter $y^{(0)}$ interval at $\underline{n}^{(0)}$)

Encoding the same amount of prior information for different prior strengths $n^{(0)}$:

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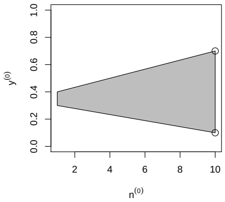

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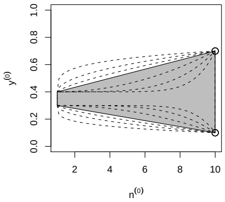

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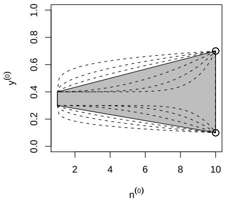

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Imprecision like confidence interval length for different pseudocounts $n^{(0)}$:

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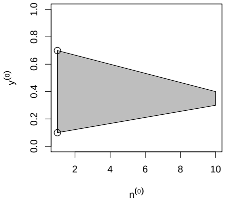

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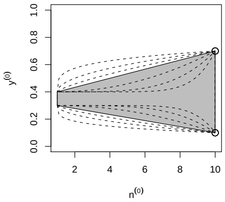

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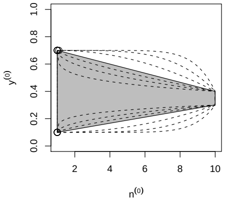
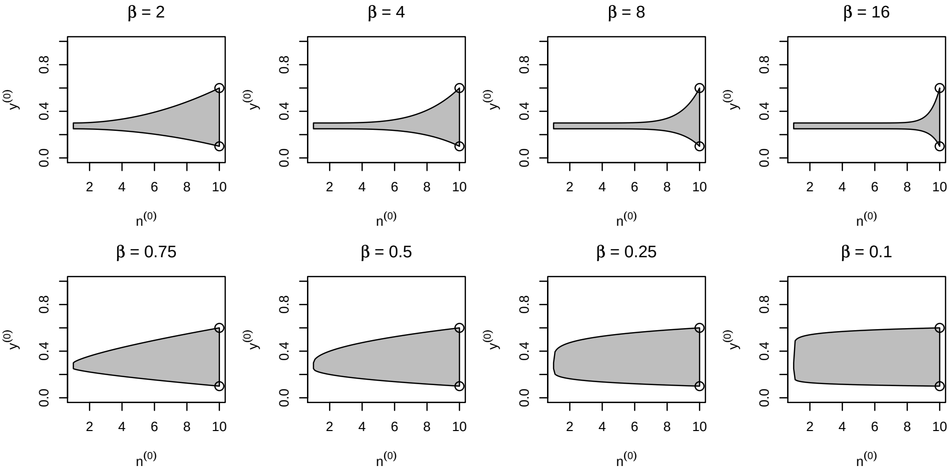
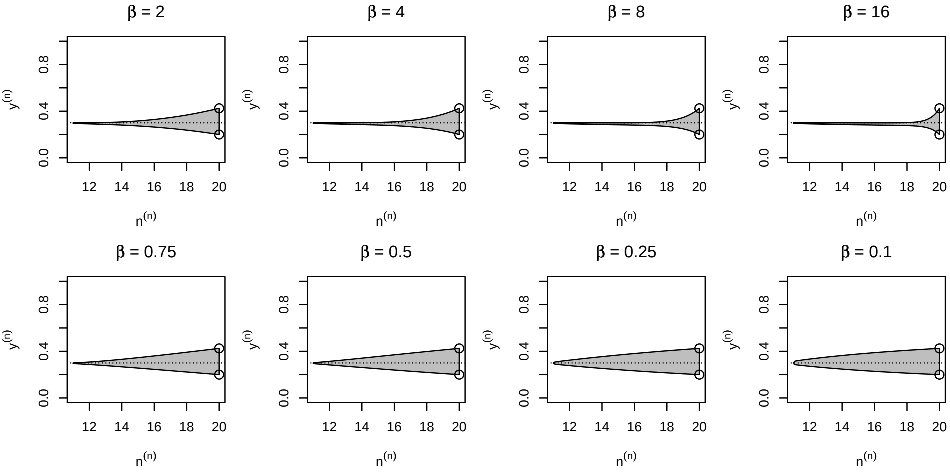


"snout left": Priors



"snout left": Posteriors when $s/n = 3/10$



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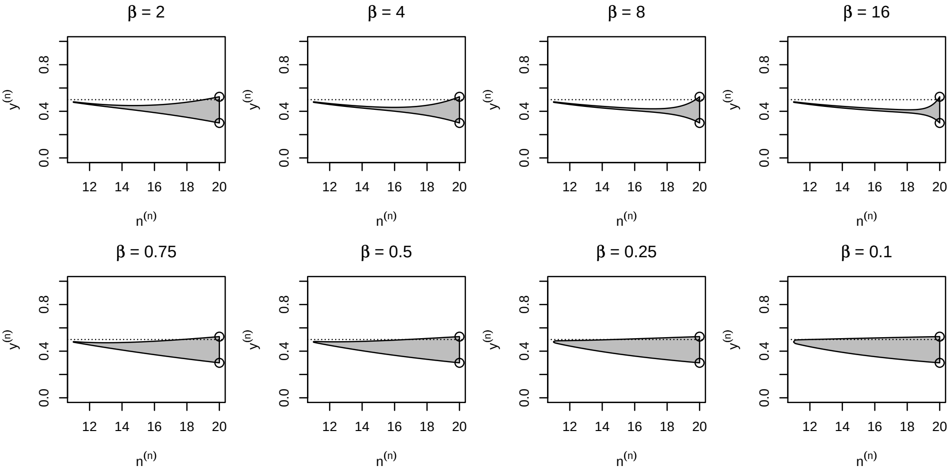

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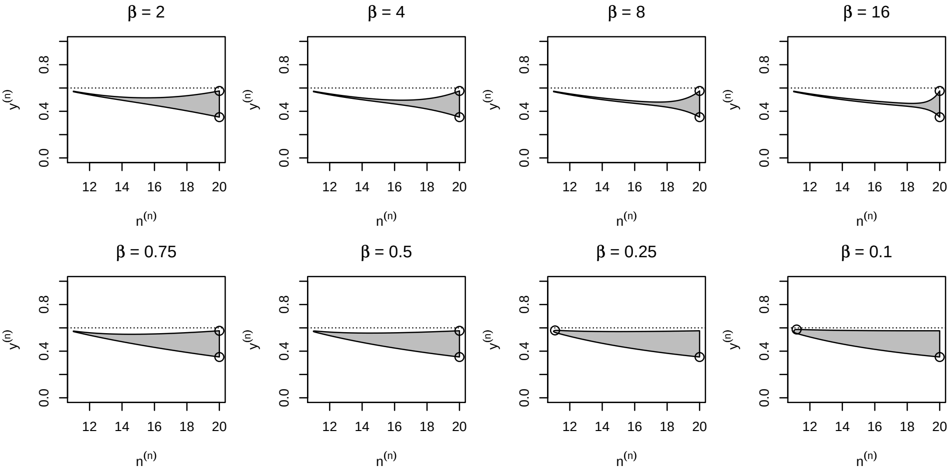

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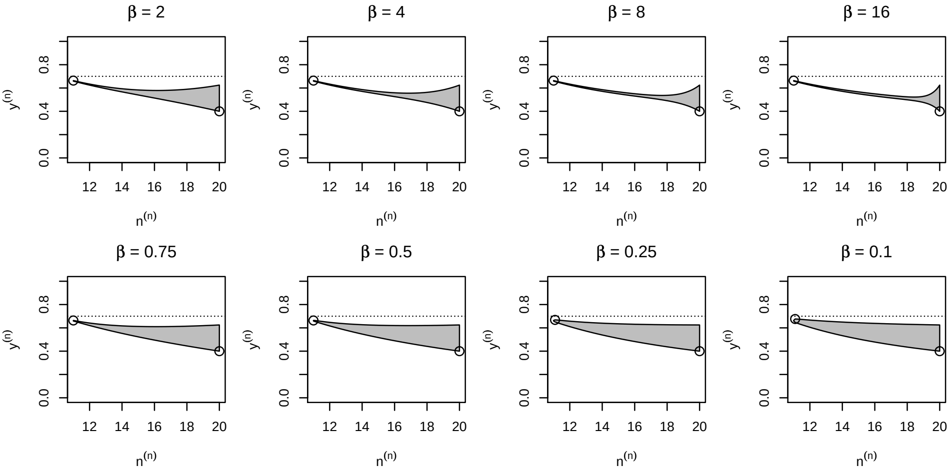

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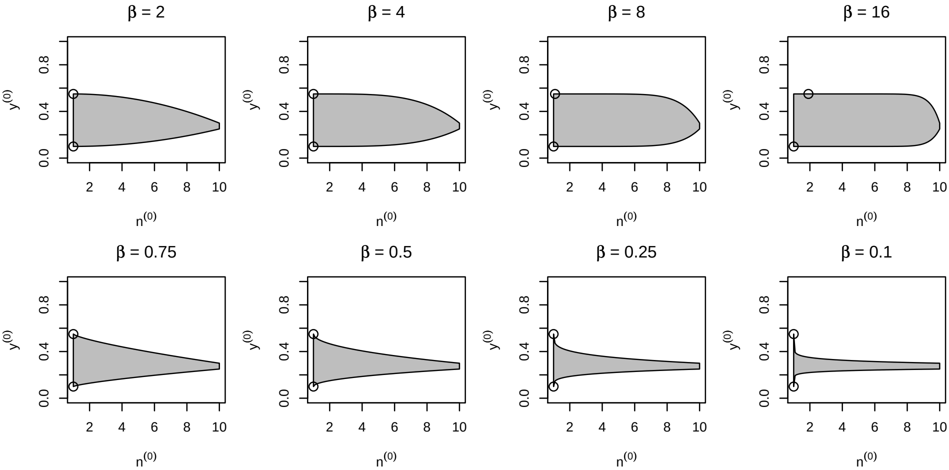

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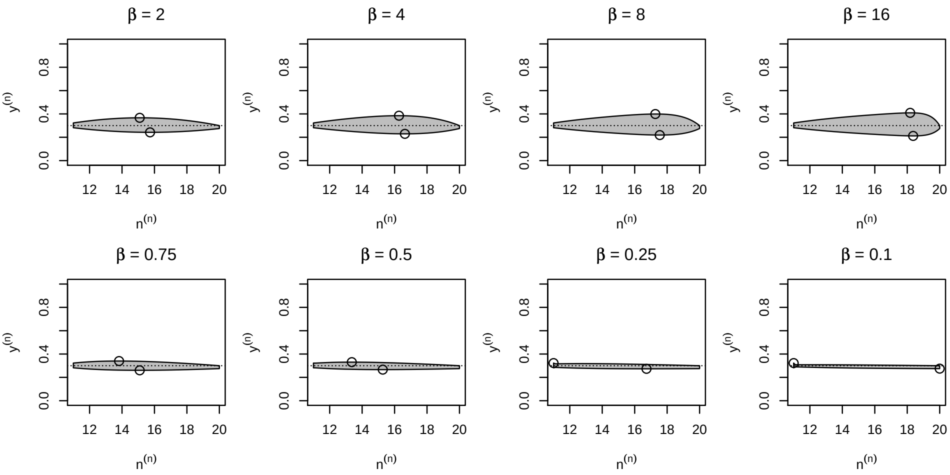

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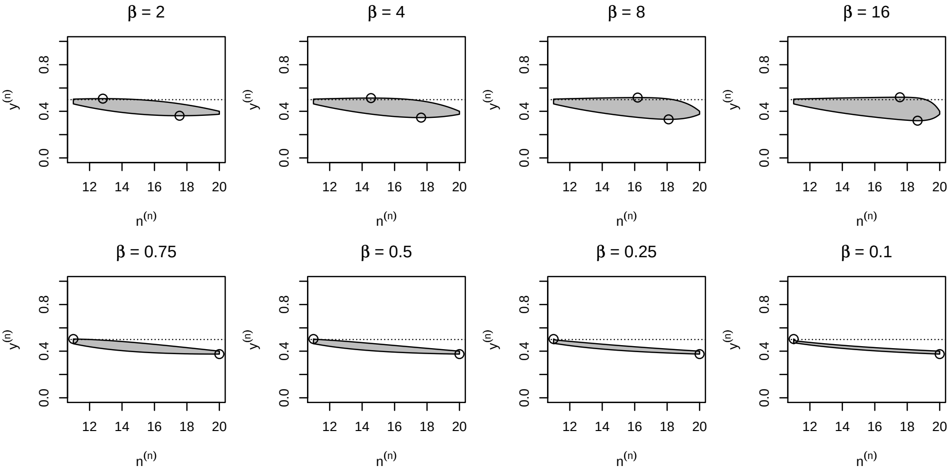

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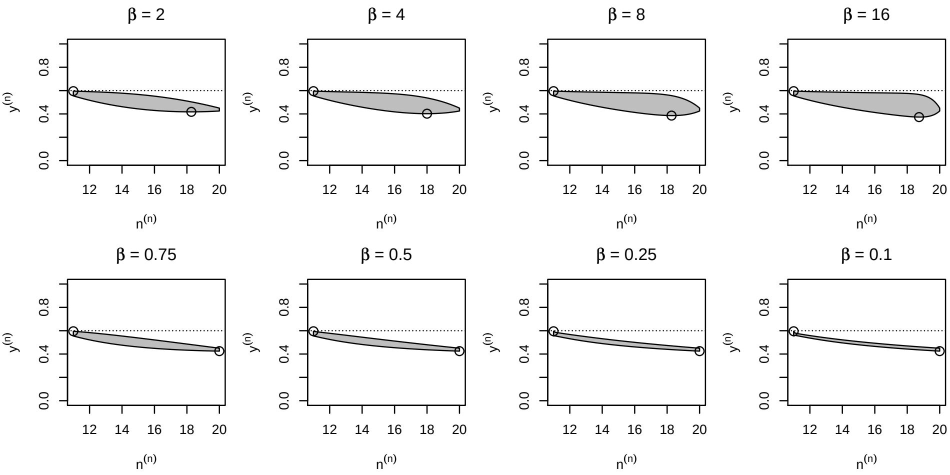

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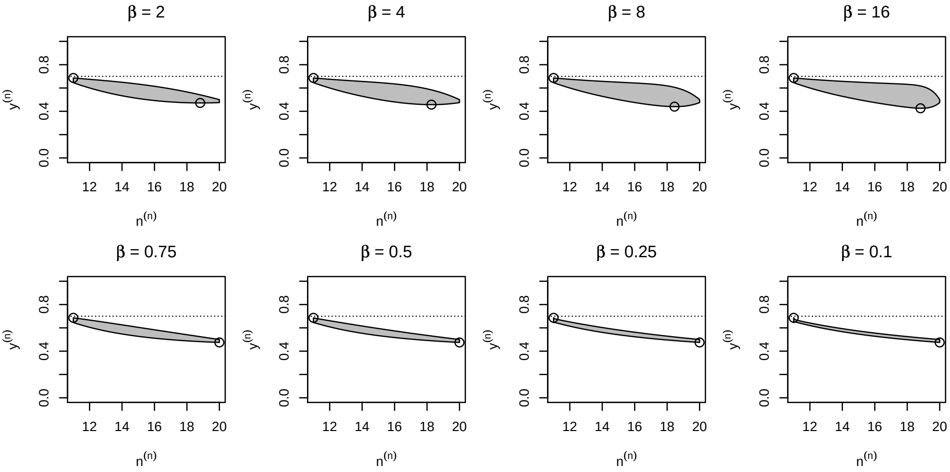

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- A shape that is invariant under updating may not exist.
- Sets of points or vertical lines are invariant shape *descriptions*, but may be impractical.
- Better focus on the aspects that matter for the inference(s) of interest: search for conjugate shape description specific to inferences, i.e. a shape description that is invariant only to those aspects that matter
  - Example: rectangle is generally a coarse approximation, but does not change inference on P

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  - $[\underline{P}_1, \overline{P}_1]$ (posterior prob. for 'Head' given one 'Head' observed)
  - $[\underline{P}_0, \overline{P}_0]$ (posterior prob. for 'Head' given one 'Tail' observed)

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- Current study of four-point-sets (ignoring all between $\underline{n}^{(0)}$ and $\overline{n}^{(0)}$) to understand which answers could lead to "snout left" and which to "snout right".