System Reliability Estimation under Prior-Data Conflict

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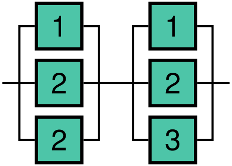
Setting: a prototype system

We want to learn about the system reliability $R_{\text{sys}}(t) = P(T_{\text{sys}} > t)$ based on



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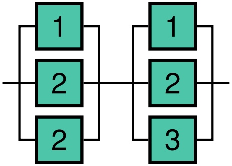

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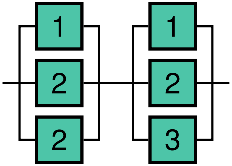

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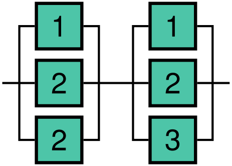

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How to combine these two information sources?

Bayesian inference

expert info + data → complete picture

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prior distribution	+	likelihood	→	posterior distribution	
$p(\lambda)$	$\times$	$p_c(\boldsymbol{t} \mid \lambda)$	$\propto$	$p(\lambda \mid \boldsymbol{t})$	► Bayes' Rule

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inverse Gamma prior		Weibull with fixed shape κ		inverse Gamma posterior ► conjugacy
$\lambda \sim \mathrm{IG}(\alpha^{(0)}, \beta^{(0)})$		$\boldsymbol{t} \mid \lambda \sim \mathrm{Wei}_\kappa(\lambda)$		$\lambda \mid \boldsymbol{t} \sim \mathrm{IG}(\alpha^{(n)}, \beta^{(n)})$

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- makes learning about component reliability tractable, just update parameters: $\alpha^{(0)} \rightarrow \alpha^{(n)}$, $\beta^{(0)} \rightarrow \beta^{(n)}$
- conjugacy holds also for censored observations
- closed form for system reliability function $R_{\mathrm{sys}}(t \mid \boldsymbol{t})$

Prior-data conflict

What if expert information and data tell different stories?

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- reparametrization helps to understand effect of prior-data conflict:

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$$n^{(n)} = n^{(0)} + n\,, \qquad y^{(n)} = \frac{n^{(0)}}{n^{(0)} + n}\, y^{(0)} + \frac{n}{n^{(0)} + n} \cdot \frac{1}{n} \textstyle\sum_{j=1}^{n} t_j^{\kappa}$$

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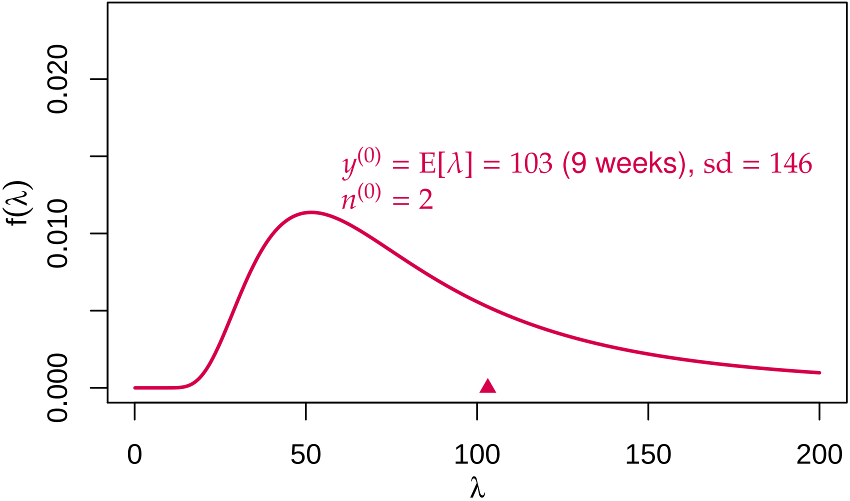
$\mathrm{E}[\lambda \mid \boldsymbol{t}]$ is a weighted average of $\mathrm{E}[\lambda]$ and $\hat{\lambda}$!

Prior-data conflict example



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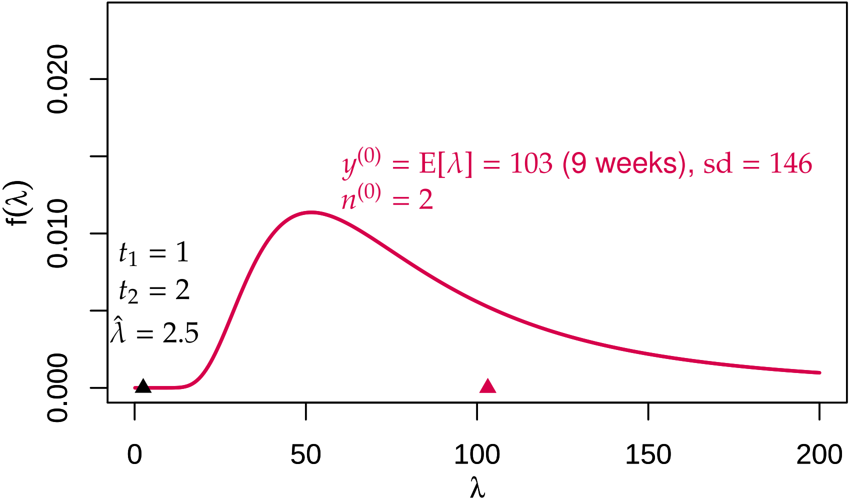

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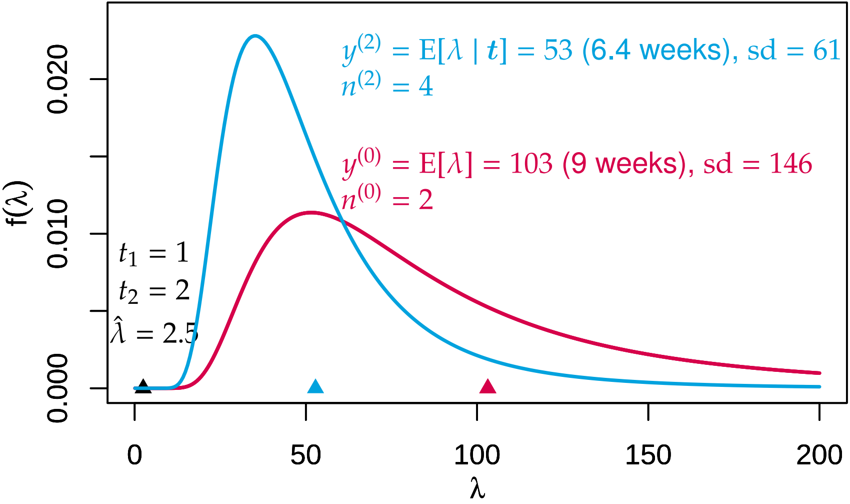

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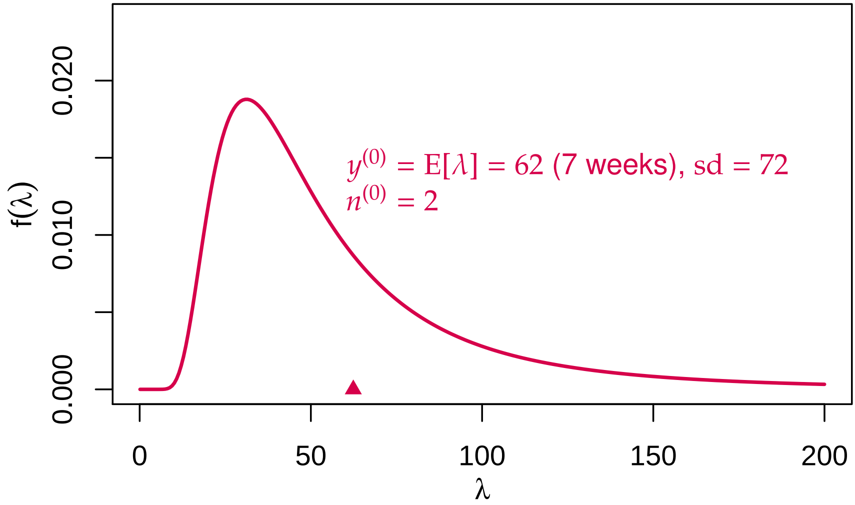

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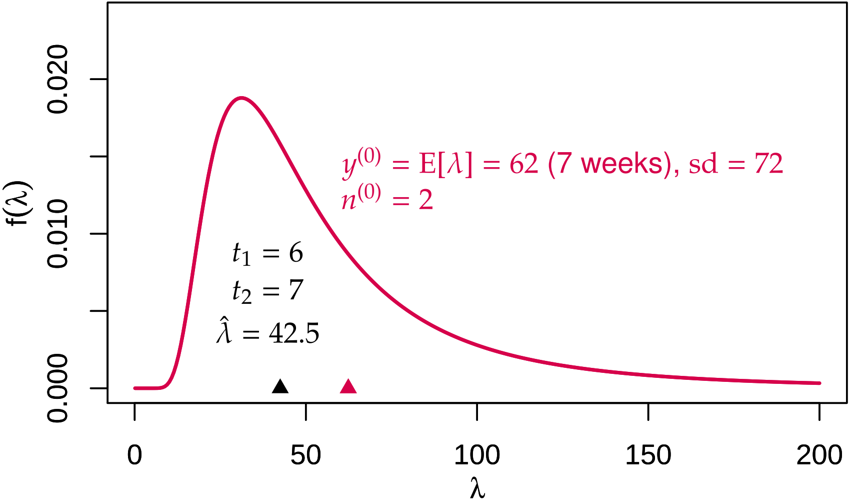

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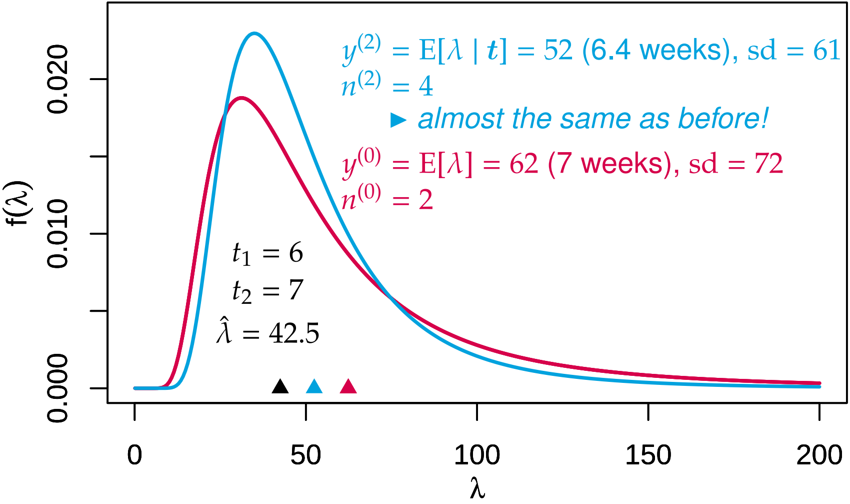

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- Can also be seen as systematic sensitivity analysis or robust Bayesian approach.

Sets of prior distributions

Uncertainty about probability statements

smaller sets = more precise probability statements

Lottery A

Number of winning tickets:
exactly known as 5 out of 100

→ $P(\text{win}) = 5/100$

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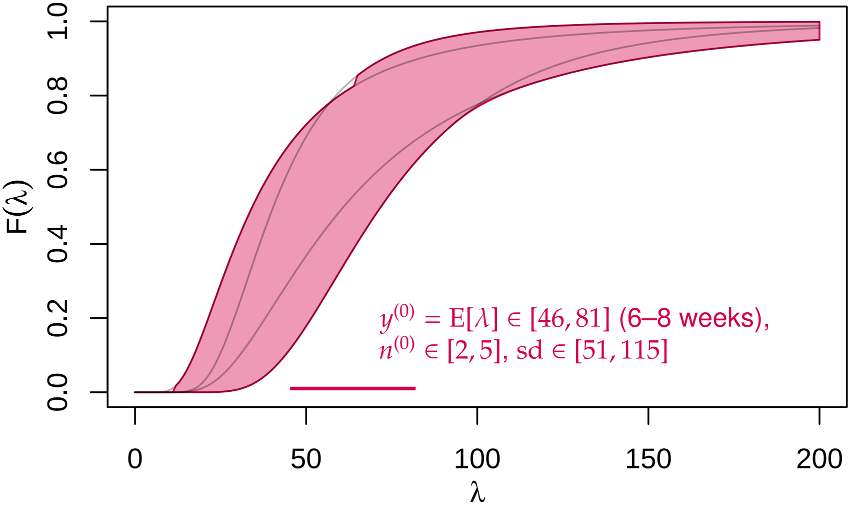
Walter and Augustin (2009), Walter (2013):
$\Pi^{(0)} = [\underline{n}^{(0)}, \overline{n}^{(0)}] \times [\underline{y}^{(0)}, \overline{y}^{(0)}]$
gives tractability & meaningful reaction to prior-data conflict:

- larger set of posteriors
- more imprecise / cautious probability statements

Sets of prior distributions: examples



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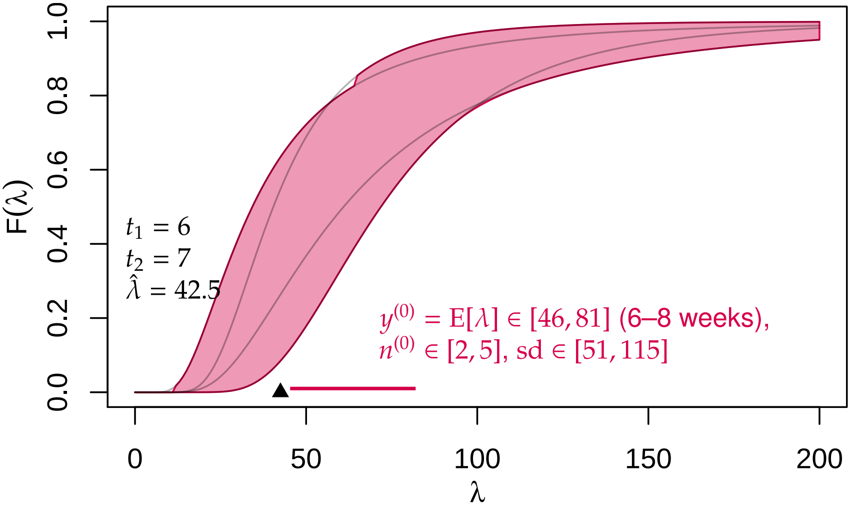

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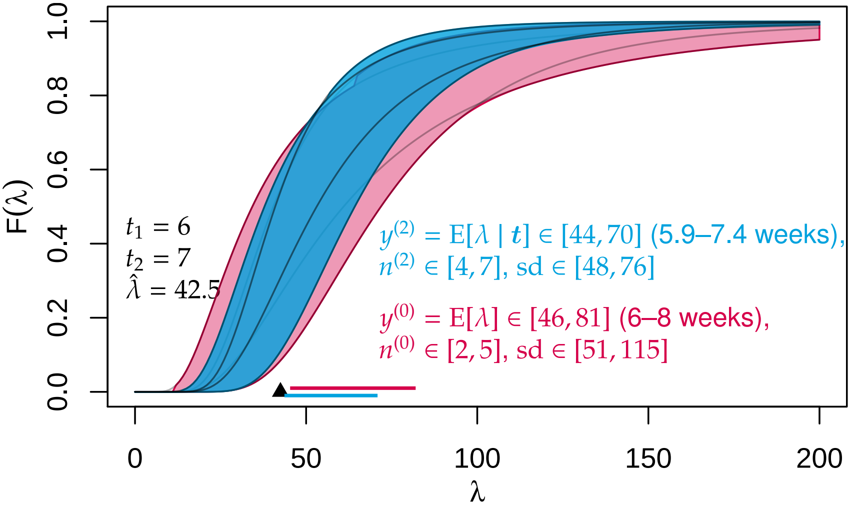

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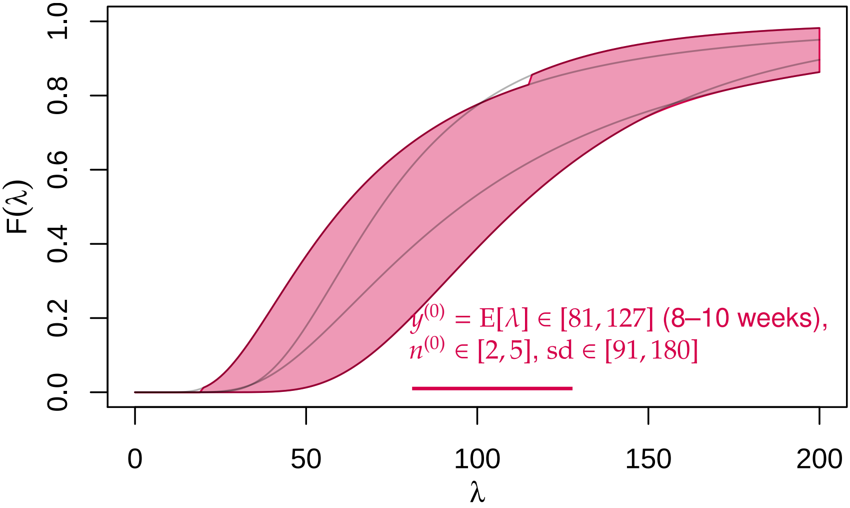

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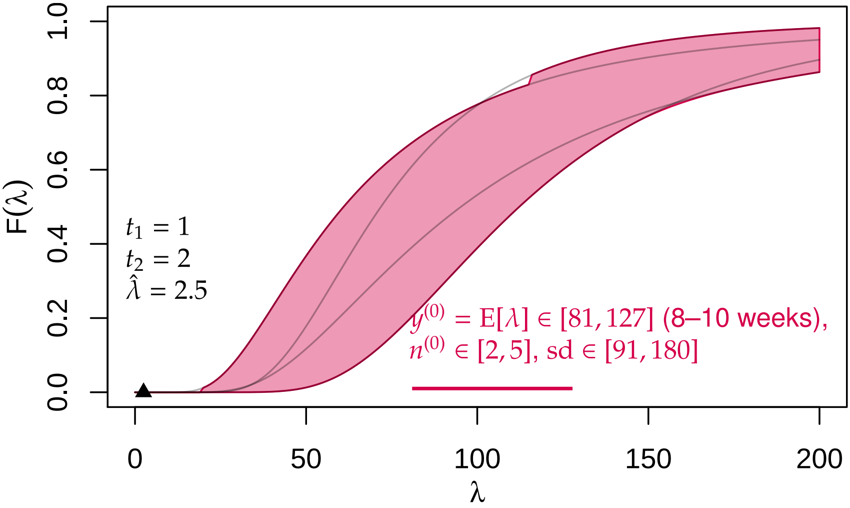

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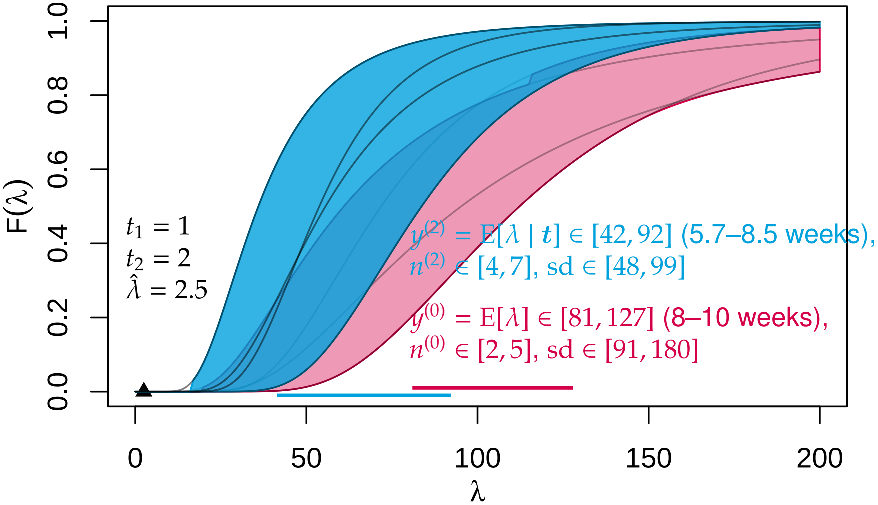

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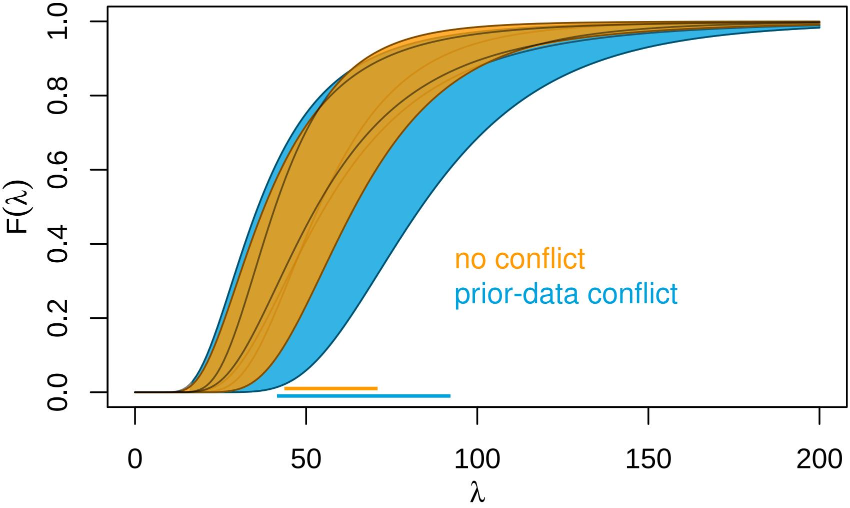

System reliability

- Closed form for the system reliability via the survival signature:

$$P\left(T_{\mathsf{sys}} > t \mid \{n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k\}^{1:K}\right) = \sum_{l_1=0}^{n_1-e_1} \cdots \sum_{l_K=0}^{n_K-e_K} \Phi(l_1,\ldots,l_K) \prod_{k=1}^{K} P(C_t^k = l_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k)$$

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Survival signature $\Phi(l_1, \ldots, l_K)$
(Coolen and Coolen-Maturi 2012)
$= P($system functions $\mid \{l_k$ k 's function$\}^{1:K})$

l_1	l_2	l_3	Φ
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0.67
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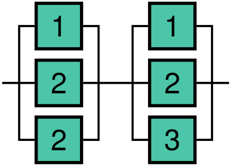

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Posterior predictive probability that l_k of the $n_k - e_k$ surviving k 's function at time t:

$$\binom{n_k-e_k}{l_k} \int [P(t > T \mid T > t_{\text{now}}, \lambda_k)]^{l_k} [P(t \leq T \mid T > t_{\text{now}}, \lambda_k)]^{n_k-e_k-l_k} f_{\lambda_k|\ldots}(\lambda_k \mid n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k)\, d\theta$$

(integral can be solved analytically)

System reliability: bounds

- Lower / upper bound through optimization for each t:

$$\underline{R}_{\textsf{sys}}\left(t \mid \{\Pi_k^{(0)}, \boldsymbol{t}^k\}^{1:K}\right) = \min_{\Pi_1^{(0)},\ldots,\Pi_K^{(0)}} P\left(T_{\textsf{sys}} > t \mid \{n_k^{(0)}, y_k^{(0)}, \boldsymbol{t}^k\}^{1:K}\right)$$

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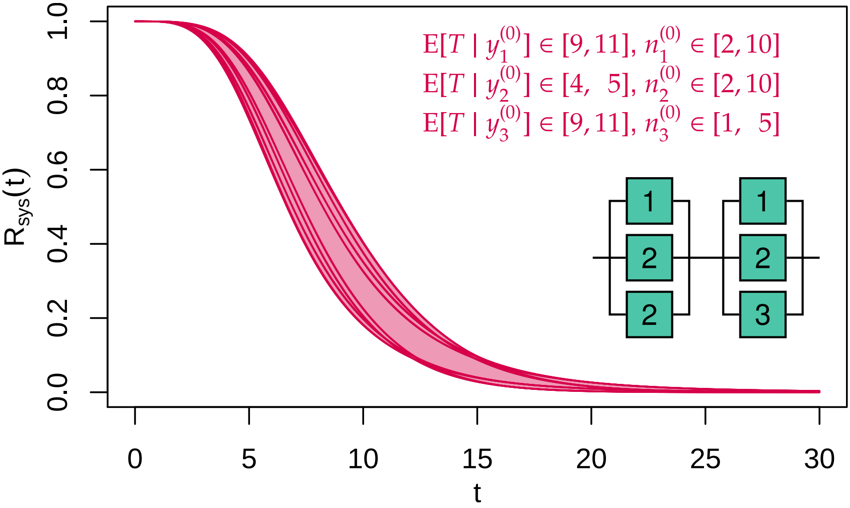

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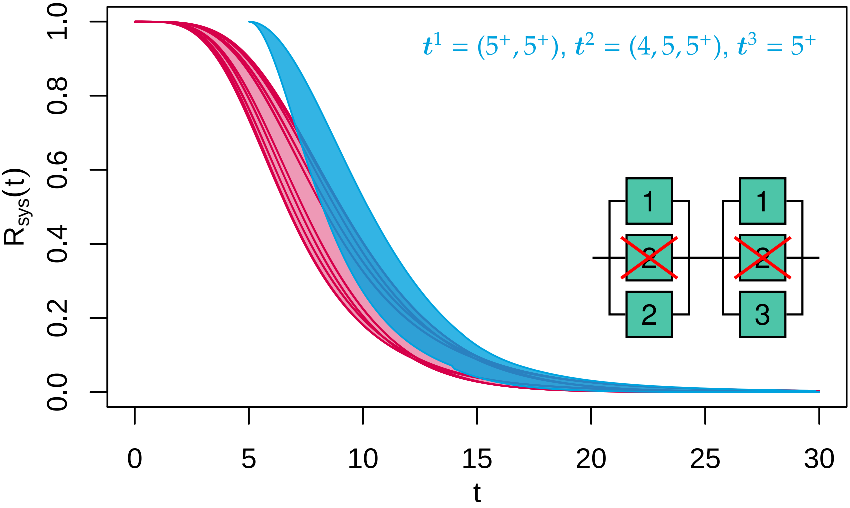

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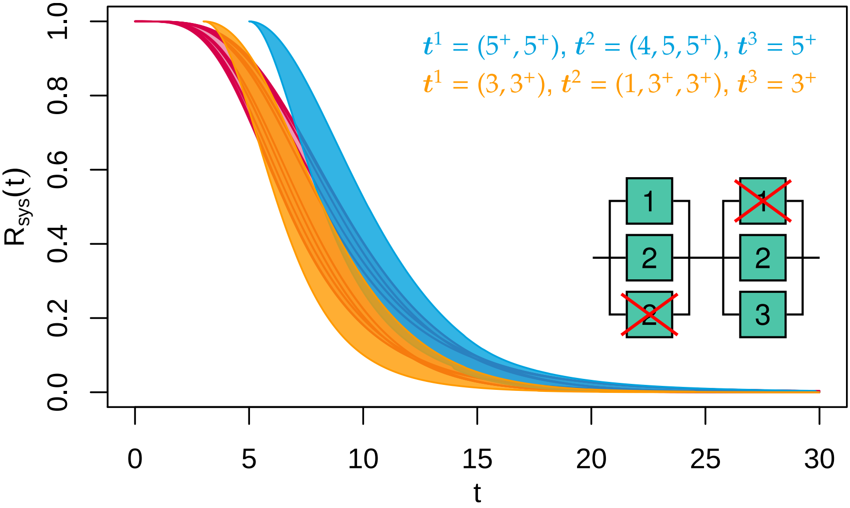

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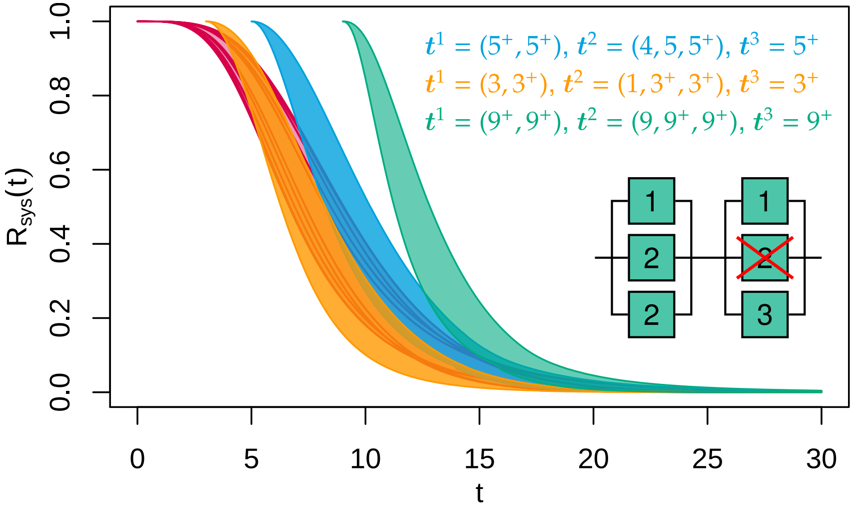

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Next steps:

- Nonparametric model
  (drop Weibull assumption for component lifetimes)
- Allow dependence between components
  (common-cause failure, ...)
- Use model for maintenance planning

References

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