# Robust Bayesian Estimation of System Reliability with Scarce and Surprising Data

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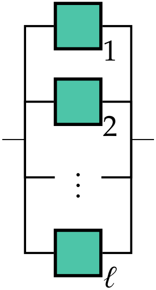
ESREL 2015

# Setting: a one of a kind parallel system



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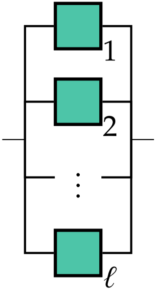


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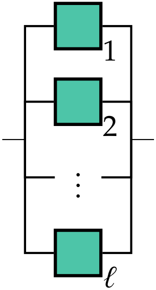


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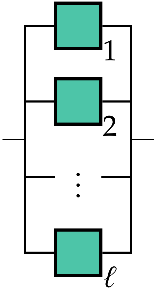


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How to combine these two information sources?

# Bayesian inference



expert info $+$ data $\rightarrow$ complete picture

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- makes learning about component reliability tractable, just update parameters: $\alpha^{(0)} \rightarrow \alpha^{(\ell)}$, $\beta^{(0)} \rightarrow \beta^{(\ell)}$
- conjugacy holds also for censored observations
- closed form for system reliability function $R_{\mathsf{sys}}(t \mid \mathbf{t})$

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What if expert information and data tell different stories?

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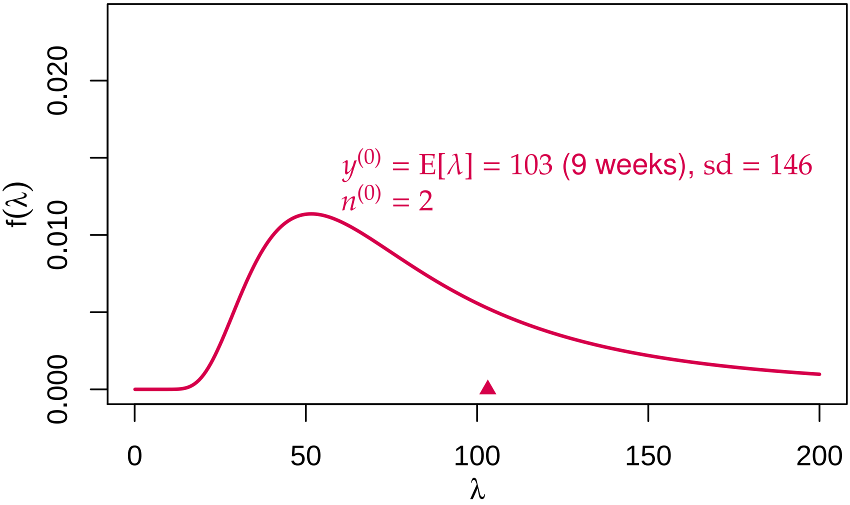
$\mathrm{E}[\lambda \mid \mathbf{t}]$ is a weighted average of $\mathrm{E}[\lambda]$ and $\hat{\lambda}$!

# Prior-data conflict example





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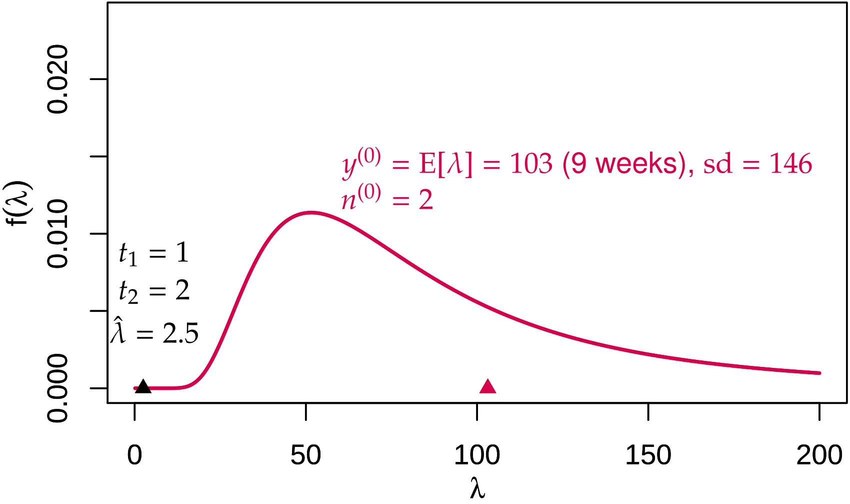

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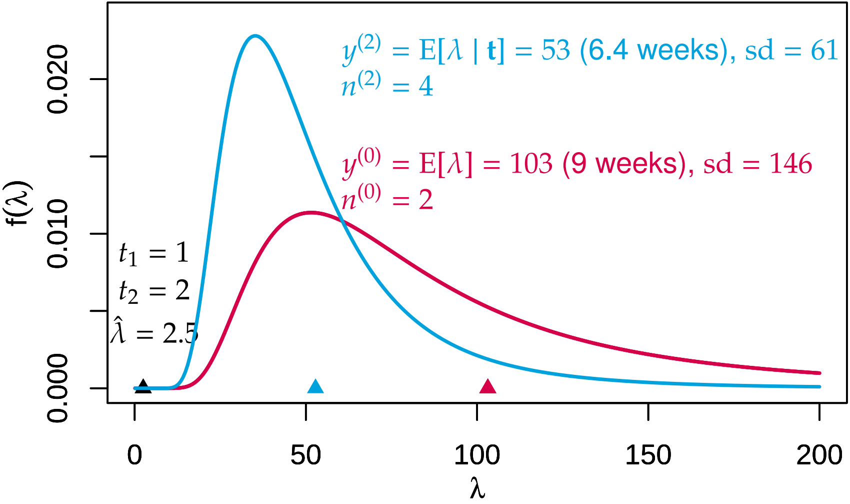

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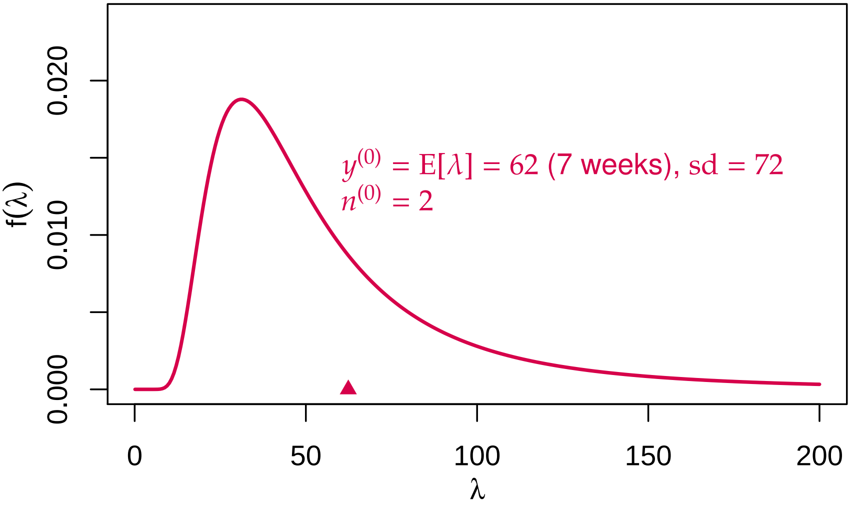




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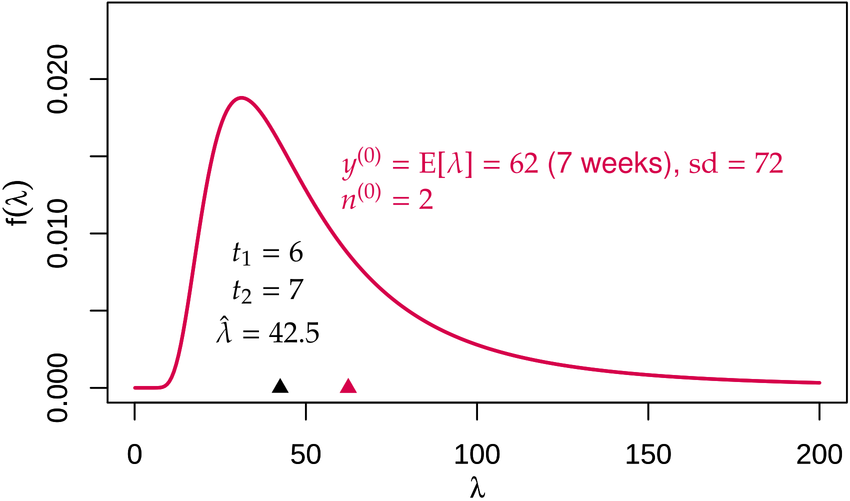

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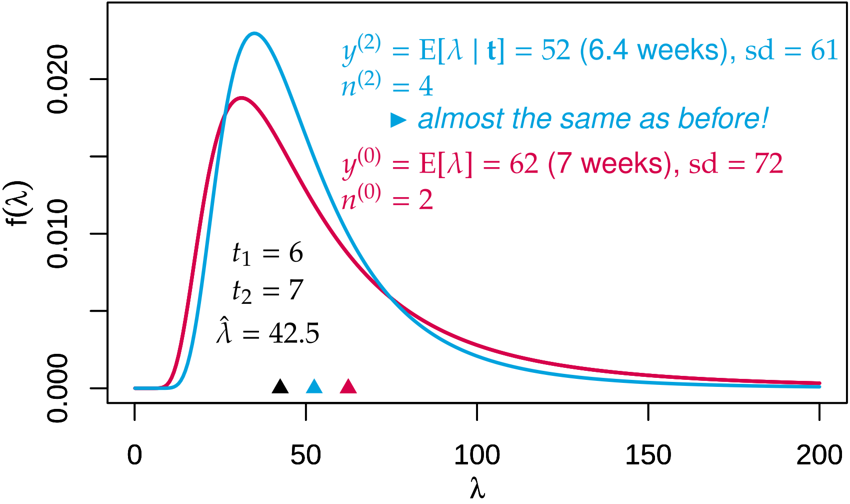

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- Can also be seen as systematic sensitivity analysis
  or robust Bayesian approach.

# Sets of prior distributions



## Uncertainty about probability statements

smaller sets = more precise probability statements

**Lottery A**

Number of winning tickets:
exactly known as 5 out of 100

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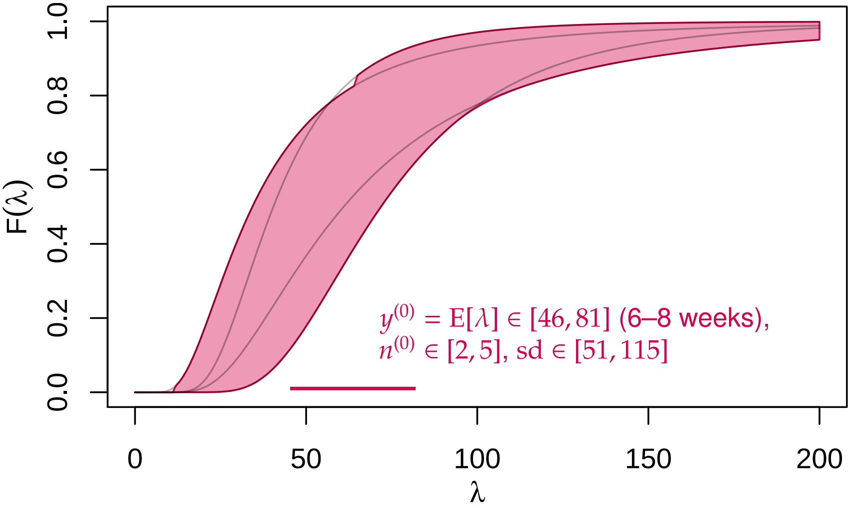
Walter and Augustin (2009), Walter (2013):
$\Pi^{(0)} = [\underline{n}^{(0)}, \overline{n}^{(0)}] \times [\underline{y}^{(0)}, \overline{y}^{(0)}]$
gives tractability & meaningful reaction to prior-data conflict:

- larger set of posteriors
- more imprecise / cautious probability statements

# Sets of prior distributions: examples





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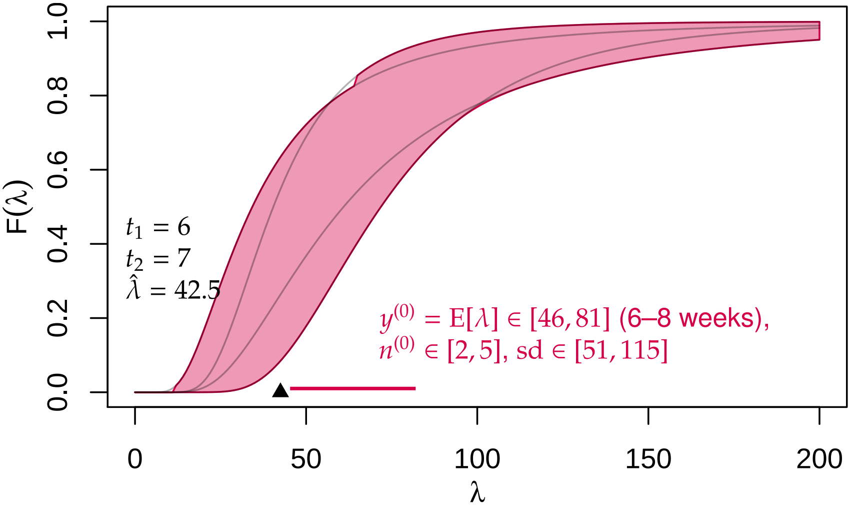

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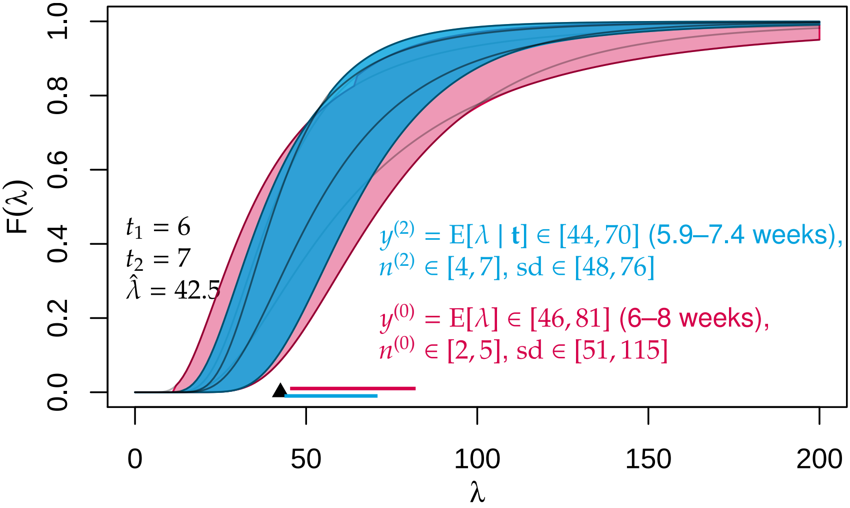

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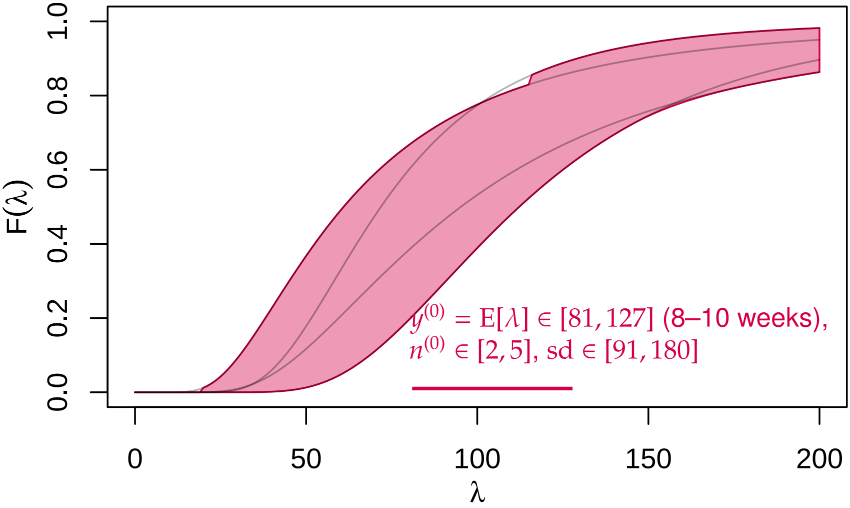

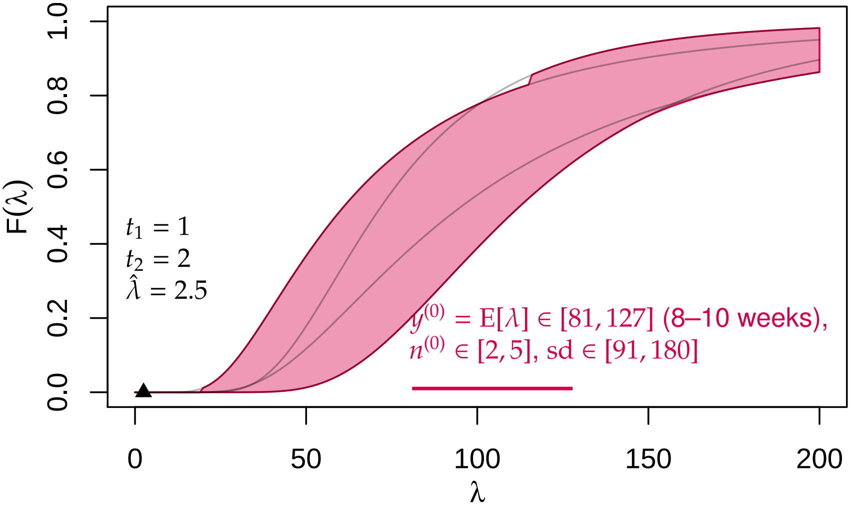
F(λ)
1.0
0.8
0.6
0.4
0.2
0.0
$t_1 = 1$
$t_2 = 2$
$\hat{\lambda} = 2.5$
$y^{(0)} = \mathrm{E}[\lambda] \in [81, 127]$ (8–10 weeks),
$n^{(0)} \in [2, 5]$, sd $\in [91, 180]$
0
50
100
150
200
λ

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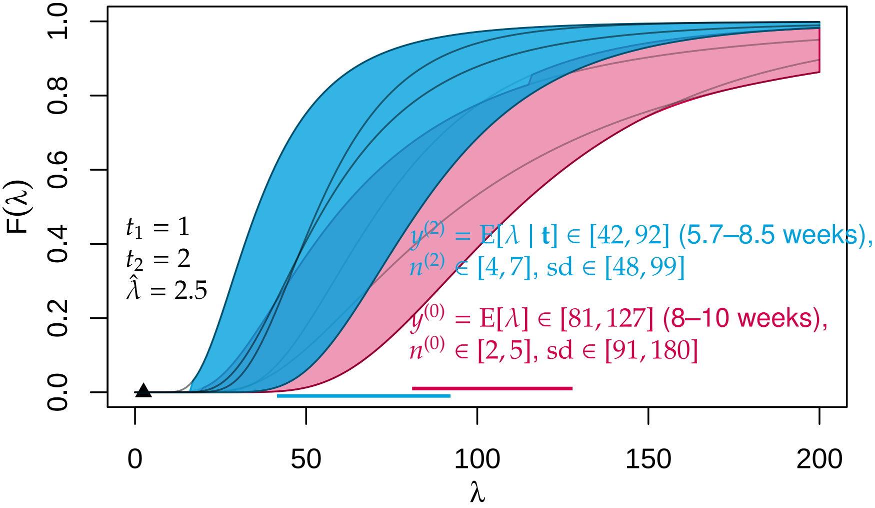

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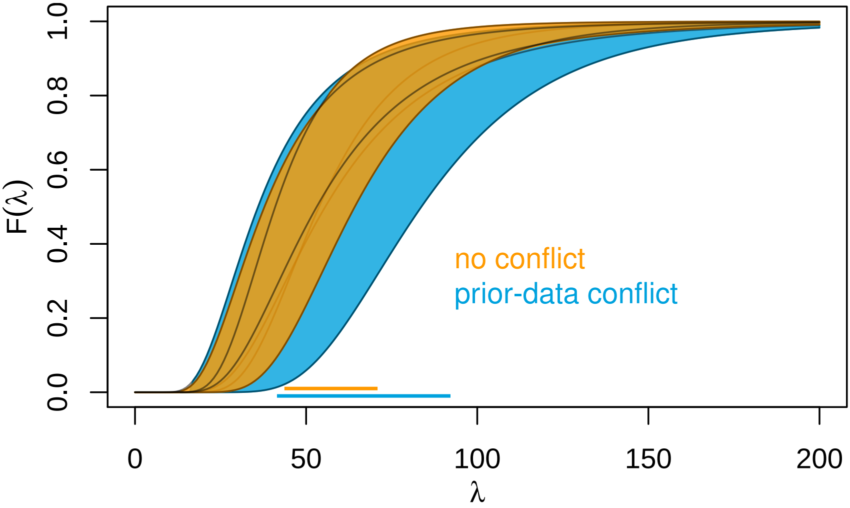

# System reliability



- Closed form for the system reliability:

$$R_{\mathsf{sys}}(t \mid \boldsymbol{t}_m^\ell, n^{(0)}, y^{(0)})$$

$$= 1 - \sum_{i=0}^{\ell-m} (-1)^i \binom{\ell - m}{i} \left( \frac{n^{(0)} y^{(0)} + (\ell - m) t_{\mathsf{now}}^k + \sum_{j=1}^m t_j^k}{n^{(0)} y^{(0)} + (\ell - m - i) t_{\mathsf{now}}^k + \sum_{j=1}^m t_j^k + i t^k} \right)^{n^{(0)}+m+1}$$

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- Lower / upper bound through optimization for each $t$:

$$\underline{R}_{\mathsf{sys}}(t \mid \boldsymbol{t}_m^\ell, \Pi^{(0)}) = \min_{n^{(0)} \in [\underline{n}^{(0)}, \overline{n}^{(0)}]} R_{\mathsf{sys}}(t \mid \boldsymbol{t}_m^\ell, n^{(0)}, \underline{y}^{(0)})$$
$$\overline{R}_{\mathsf{sys}}(t \mid \boldsymbol{t}_m^\ell, \Pi^{(0)}) = \max_{n^{(0)} \in [\underline{n}^{(0)}, \overline{n}^{(0)}]} R_{\mathsf{sys}}(t \mid \boldsymbol{t}_m^\ell, n^{(0)}, \overline{y}^{(0)})$$

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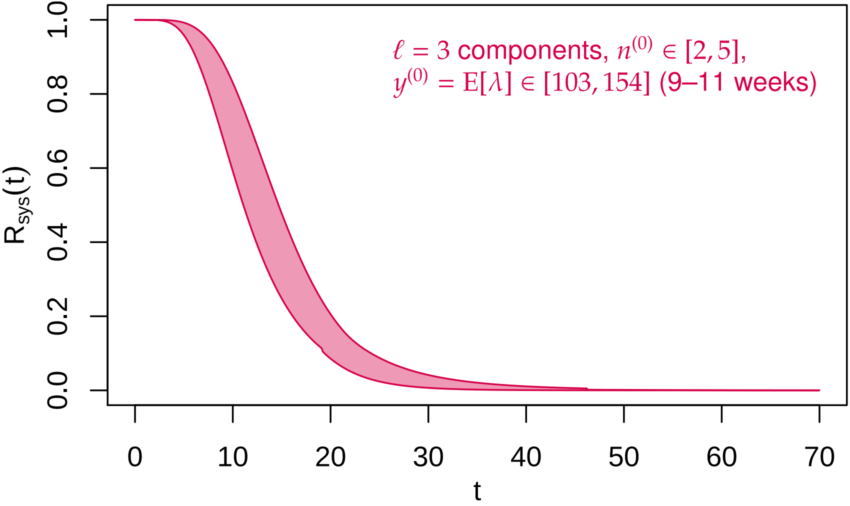


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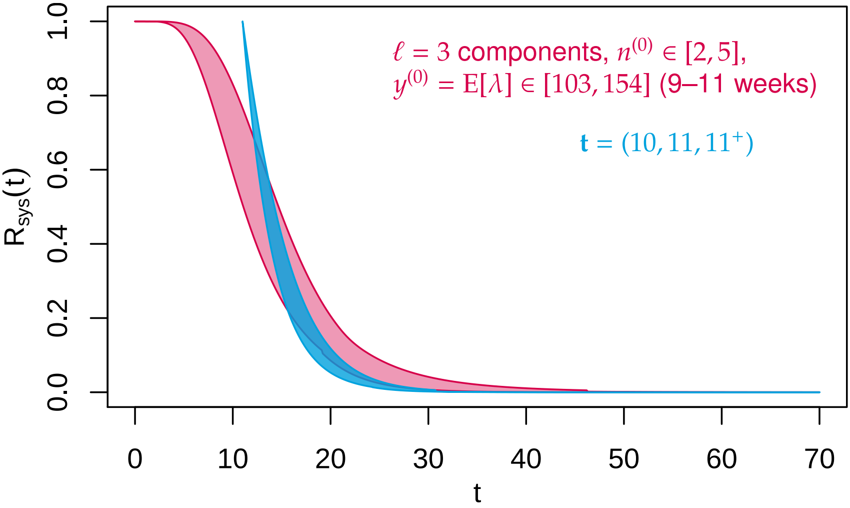

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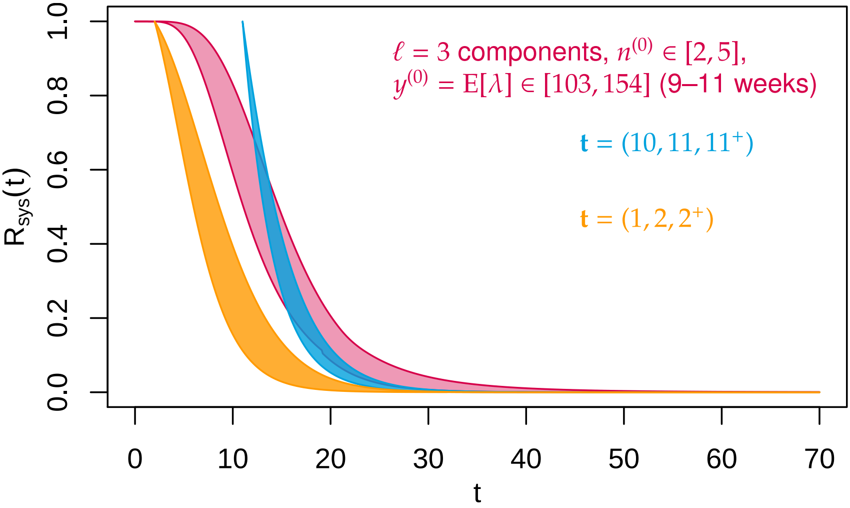

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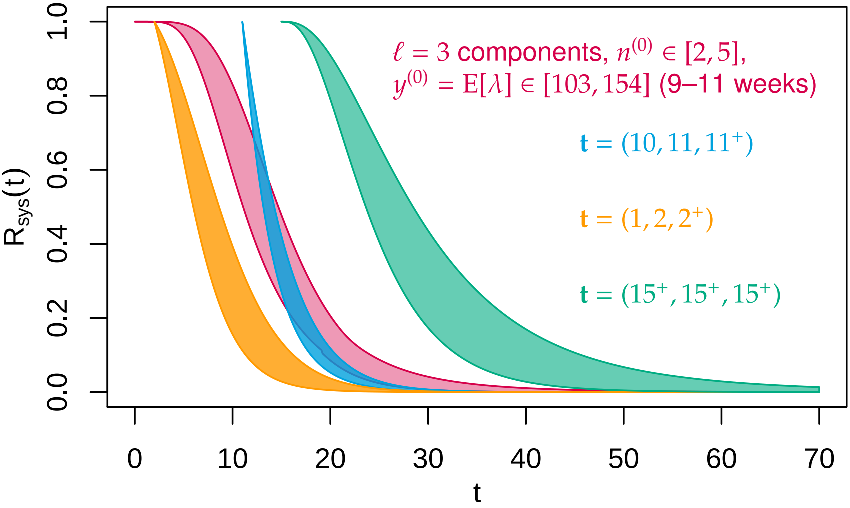

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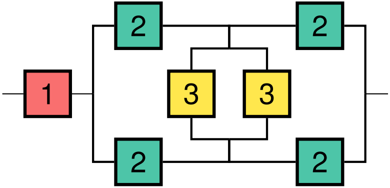

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- Allow dependence between components (common-cause failure, . . . )
- Use model for maintenance planning



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# References



Coolen, Frank P. A. and Tahani Coolen-Maturi (2012). “Generalizing the Signature to Systems with Multiple Types of Components”. In: *Complex Systems and Dependability*. Ed. by W. Zamojski et al. Vol. 170. Advances in Intelligent and Soft Computing. Springer, pp. 115–130. DOI: 10.1007/978-3-642-30662-4_8.

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Walter, Gero and Thomas Augustin (2009). “Imprecision and Prior-Data Conflict in Generalized Bayesian Inference”. In: *Journal of Statistical Theory and Practice* 3, pp. 255–271. DOI: 10.1080/15598608.2009.10411924.

# General systems via the survival signature



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## System Reliability Estimation under Prior-Data Conflict

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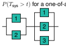
Durham University

DINALOG Dutch Institute for Advanced Logistics

### System reliability

We want to find the *system reliability* $P(T_{\text{sys}} > t)$ for a one-of-a-kind system:



$n_1 = 2$, $n_2 = 2$, $n_3 = 1$

The system consists of $n_k$ exchangeable components of types 1, …, K.

### Component Lifetimes

The lifetime for each k is assumed as Weibull with fixed shape $\beta$:

$$F_k(t \mid \lambda_k) = 1 - e^{-\frac{t^\beta}{\lambda_k}}$$
$$\mathrm{E}[T \mid \lambda_k] = \sqrt[\beta]{\lambda_k}\,\Gamma(1 + 1/\beta)$$

We have information on $\lambda_k$ from the component manufacturer, but do not fully trust it and model knowledge on $\lambda_k$ cautiously with a *set of priors* $\mathcal{M}_k^{(0)}$.

### Set of Priors

Each $\mathcal{M}_k^{(0)}$ is taken as a set of conjugate inverse Gamma priors. In terms of canonical parameters $n^{(0)}$, $y^{(0)}$, $\mathcal{M}_k^{(0)} = \{ \mathrm{IG}(n_k^{(0)} + 1, n_k^{(0)} y_k^{(0)}) \mid [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}] \}$, where $y_k^{(0)} = \mathrm{E}[\lambda_k \mid n_k^{(0)}, y_k^{(0)}]$ and $n_k^{(0)}$ = pseudocounts.
The prior parameter set $\Pi_k^{(0)} = [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}]$ allows for more imprecision in case of *prior-data conflict* [2].

### Data

We observe the system from startup until $t_{\text{now}}$. For each $k$, the data $\mathbf{t}^k_{e_k;n_k}$ consists of $e_k$ failure times and $n_k - e_k$ censored observations.
$n_k^{(0)}$ and $y_k^{(0)}$ are updated to $n_k^{(n)}$ and $y_k^{(n)}$ via Bayes' Rule.

Need to minimize over $n_k^{(0)}$'s only, as min must be reached for $\underline{y}_k^{(0)}$'s (lower expected lifetimes = lower component survival probabilities = lower system survival probability).

$$\underline{P}\left(T_{\text{sys}} > t \mid \{n_k^{(0)}, y_k^{(0)}, \mathbf{t}^k_{e_k;n_k}\}^{1:K}\right) = \min_{n_1^{(0)},\ldots,n_K^{(0)}} \sum_{l_1=0}^{n_1-e_1} \cdots \sum_{l_K=0}^{n_K-e_K} \Phi(l_1,\ldots,l_K) \prod_{k=1}^{K} P(C_t^k = l_k \mid n_k^{(0)}, \underline{y}_k^{(0)}, \mathbf{t}^k_{e_k;n_k})$$

Survival signature $\Phi(l_1,\ldots,l_K)$ [1]
$= P(\text{system functions} \mid \{l_k \text{ k's function}\}^{1:K})$

| $l_1$ | $l_2$ | $l_3$ | $\Phi$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0.5 |
| 0 | 2 | 0 | 1 |

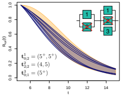
| $l_1$ | $l_2$ | $l_3$ | $\Phi$ |
|---|---|---|---|
| 0 | 2 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0.5 |
| 1 | 1 | 1 | 0.75 |
| ⋮ | ⋮ | ⋮ | ⋮ |

Posterior predictive probability that $l_k$ of the $n_k - e_k$ surviving k's function at time $t$:

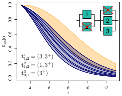
$$\binom{n_k - e_k}{l_k} \int \left[P_k(T > t \mid T > t_{\text{now}}, \lambda_k)\right]^{l_k} \times \left[1 - P_k(T > t \mid T > t_{\text{now}}, \lambda_k)\right]^{n_k - e_k - l_k} f_{\lambda_k|\ldots}(\lambda_k \mid n_k^{(0)}, y_k^{(0)}, \mathbf{t}^k_{e_k;n_k})\, \mathrm{d}\lambda_k$$
$$= \binom{n_k - e_k}{l_k} \sum_{j=0}^{n_k - e_k - l_k} (-1)^j \binom{n_k - e_k - l_k}{j} \left( \frac{n_k^{(n)} y_k^{(n)}}{n_k^{(n)} y_k^{(n)} + (l_k + j)(t^\beta - (t_{\text{now}})^\beta)} \right)^{n_k^{(n)}+1}$$

We assume $\beta = 2$, $\mathrm{E}[T \mid y_1^{(0)}] \in [9, 11]$, $n_1^{(0)} \in [2, 10]$, $\mathrm{E}[T \mid y_2^{(0)}] \in [4, 5]$, $n_2^{(0)} \in [8, 16]$, and $\mathrm{E}[T \mid y_3^{(0)}] \in [9, 11]$, $n_3^{(0)} \in [1, 5]$.

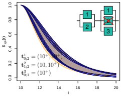
### Failure times as expected




### Surprisingly early failures




### Surprisingly late failures




### References

[1] Frank P. A. Coolen and Tahani Coolen-Maturi. Generalizing the signature to systems with multiple types of components. In W. Zamojski, J. Mazurkiewicz, J. Sugier, T. Walkowiak, and J. Kacprzyk, editors, *Complex Systems and Dependability*, volume 170 of *Advances in Intelligent and Soft Computing*, pages 115–130. Springer, 2012.
[2] G. Walter. *Generalized Bayesian Inference under Prior-Data Conflict*. PhD thesis, Department of Statistics, LMU Munich, 2013. http://edoc.ub.uni-muenchen.de/17059.

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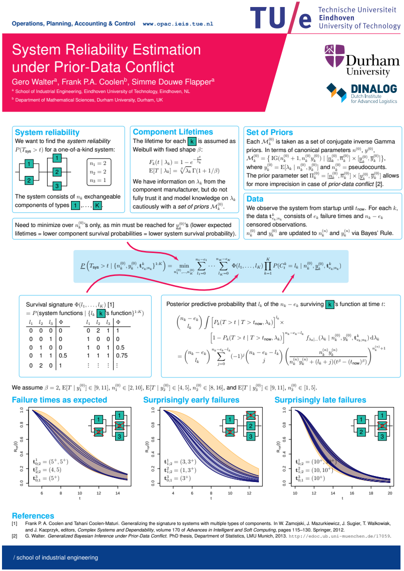
# General systems via the survival signature





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Technische Universiteit Eindhoven University of Technology

## System Reliability Estimation under Prior-Data Conflict

Gero Walter[a], Frank P.A. Coolen[b], Simme Douwe Flapper[a]

[a] School of Industrial Engineering, Eindhoven University of Technology, Eindhoven, NL

[b] Department of Mathematical Sciences, Durham University, Durham, UK

Durham University

DINALOG Dutch Institute for Advanced Logistics

### System reliability

We want to find the *system reliability* $P(T_{sys} > t)$ for a one-of-a-kind system:

$n_1 = 2$, $n_2 = 2$, $n_3 = 1$

The system consists of $n_k$ exchangeable components of types 1, …, K.

### Component Lifetimes

The lifetime for each k is assumed as Weibull with fixed shape $\beta$:

$$F_k(t \mid \lambda_k) = 1 - e^{-\frac{t^\beta}{\lambda_k}}$$
$$\mathrm{E}[T \mid \lambda_k] = \sqrt[\beta]{\lambda_k}\,\Gamma(1 + 1/\beta)$$

We have information on $\lambda_k$ from the component manufacturer, but do not fully trust it and model knowledge on $\lambda_k$ cautiously with a *set of priors* $\mathcal{M}_k^{(0)}$.

### Set of Priors

Each $\mathcal{M}_k^{(0)}$ is taken as a set of conjugate inverse Gamma priors. In terms of canonical parameters $n^{(0)}$, $y^{(0)}$, $\mathcal{M}_k^{(0)} = \{ \mathrm{IG}(n_k^{(0)} + 1, n_k^{(0)} y_k^{(0)}) \mid [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}] \}$, where $y_k^{(0)} = \mathrm{E}[\lambda_k \mid n_k^{(0)}, y_k^{(0)}]$ and $n_k^{(0)}$ = pseudocounts. The prior parameter set $\Pi_k^{(0)} = [\underline{n}_k^{(0)}, \overline{n}_k^{(0)}] \times [\underline{y}_k^{(0)}, \overline{y}_k^{(0)}]$ allows for more imprecision in case of *prior-data conflict* [2].

### Data

We observe the system from startup until $t_{now}$. For each $k$, the data $\mathbf{t}^k_{e_k;n_k}$ consists of $e_k$ failure times and $n_k - e_k$ censored observations. $n_k^{(0)}$ and $y_k^{(0)}$ are updated to $n_k^{(n)}$ and $y_k^{(n)}$ via Bayes' Rule.

Need to minimize over $n_k^{(0)}$'s only, as min must be reached for $\underline{y}_k^{(0)}$'s (lower expected lifetimes = lower component survival probabilities = lower system survival probability).

$$\underline{P}\left(T_{sys} > t \mid \{n_k^{(0)}, y_k^{(0)}, \mathbf{t}^k_{e_k;n_k}\}^{1:K}\right) = \min_{n_1^{(0)},\ldots,n_K^{(0)}} \sum_{l_1=0}^{n_1-e_1} \cdots \sum_{l_K=0}^{n_K-e_K} \Phi(l_1,\ldots,l_K) \prod_{k=1}^{K} P(C_t^k = l_k \mid n_k^{(0)}, \underline{y}_k^{(0)}, \mathbf{t}^k_{e_k;n_k})$$

Survival signature $\Phi(l_1, \ldots, l_K)$ [1] $= P$(system functions $\mid \{l_k$ k's function$\}^{1:K})$

| $l_1$ | $l_2$ | $l_3$ | $\Phi$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0.5 |
| 0 | 2 | 0 | 1 |
| 0 | 2 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0.5 |
| 1 | 1 | 1 | 0.75 |
| ⋮ | ⋮ | ⋮ | ⋮ |

Posterior predictive probability that $l_k$ of the $n_k - e_k$ surviving k's function at time $t$:

$$\binom{n_k - e_k}{l_k} \int \left[P_k(T > t \mid T > t_{now}, \lambda_k)\right]^{l_k} \times \left[1 - P_k(T > t \mid T > t_{now}, \lambda_k)\right]^{n_k - e_k - l_k} f_{\lambda_k \mid \ldots}(\lambda_k \mid n_k^{(0)}, y_k^{(0)}, \mathbf{t}^k_{e_k;n_k})\, d\lambda_k$$
$$= \binom{n_k - e_k}{l_k} \sum_{j=0}^{n_k - e_k - l_k} (-1)^j \binom{n_k - e_k - l_k}{j} \left( \frac{n_k^{(n)} y_k^{(n)}}{n_k^{(n)} y_k^{(n)} + (l_k + j)(t^\beta - (t_{now})^\beta)} \right)^{n_k^{(n)}+1}$$

We assume $\beta = 2$, $\mathrm{E}[T \mid y_1^{(0)}] \in [9, 11]$, $n_1^{(0)} \in [2, 10]$, $\mathrm{E}[T \mid y_2^{(0)}] \in [4, 5]$, $n_2^{(0)} \in [8, 16]$, and $\mathrm{E}[T \mid y_3^{(0)}] \in [9, 11]$, $n_3^{(0)} \in [1, 5]$.

### Failure times as expected

$\mathbf{t}^1_{0;2} = (5^+, 5^+)$, $\mathbf{t}^2_{2;2} = (4, 5)$, $\mathbf{t}^3_{0;1} = (5^+)$

### Surprisingly early failures

$\mathbf{t}^1_{1;2} = (3, 3^+)$, $\mathbf{t}^2_{1;2} = (1, 3^+)$, $\mathbf{t}^3_{0;1} = (3^+)$

### Surprisingly late failures

$\mathbf{t}^1_{0;2} = (10^+, 10^+)$, $\mathbf{t}^2_{1;2} = (10, 10^+)$, $\mathbf{t}^3_{0;1} = (10^+)$

### References

[1] Frank P. A. Coolen and Tahani Coolen-Maturi. Generalizing the signature to systems with multiple types of components. In W. Zamojski, J. Mazurkiewicz, J. Sugier, T. Walkowiak, and J. Kacprzyk, editors, *Complex Systems and Dependability*, volume 170 of *Advances in Intelligent and Soft Computing*, pages 115–130. Springer, 2012.

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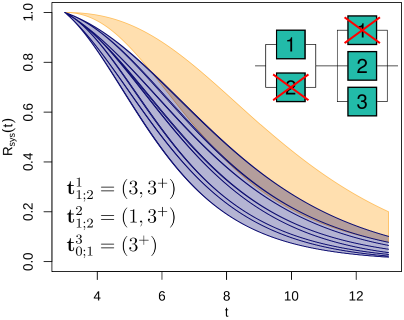
## Surprisingly early failures



# General systems via the survival signature

Survival signature $\Phi(l_1, \ldots, l_K)$ [1]
$= P($system functions $\mid \{l_k$ k 's function$\}^{1:K})$

| $l_1$ | $l_2$ | $l_3$ | $\Phi$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0.5 |
| 0 | 2 | 0 | 1 |

| $l_1$ | $l_2$ | $l_3$ | $\Phi$ |
|---|---|---|---|
| 0 | 2 | 1 | 1 |
| 1 | 0 | 0 | 0 |
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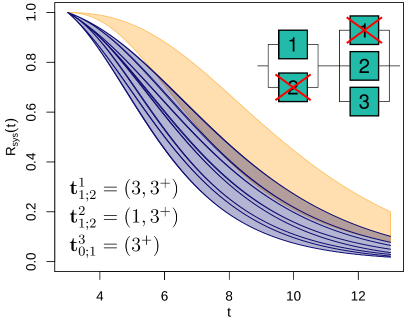
| 1 | 1 | 1 | 0.75 |
| ⋮ | ⋮ | ⋮ | ⋮ |

## Surprisingly early failures



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$= P$(system functions | $\{l_k$ k 's function$\}^{1:K}$)

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|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 0.5 |
| 0 | 2 | 0 | 1 |

| $l_1$ | $l_2$ | $l_3$ | $\Phi$ |
|---|---|---|---|
| 0 | 2 | 1 | 1 |
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| ⋮ | ⋮ | ⋮ | ⋮ |

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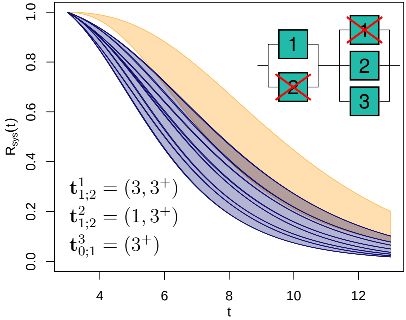


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