# The Effect of Prior-Data Conflict in Bayesian Linear Regression

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# Prior-Data Conflict

## Prior-Data Conflict $\hat{=}$ situation in which…

- …*informative prior beliefs* and *trusted data* (sampling model correct, no outliers, etc.) are in conflict
- "… the prior [places] its mass primarily on distributions in the sampling model for which the observed data is surprising." (Evans & Moshonov, 2006)

# Simple Example: Dirichlet-Multinomial-Model

$$
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\text{Data:} & \mathbf{k} & \sim & \mathrm{M}(\boldsymbol{\theta}) \\
\text{conjugate prior:} & \boldsymbol{\theta} & \sim & \mathrm{Dir}(\boldsymbol{\alpha}) \\
\hline
\text{posterior:} & \boldsymbol{\theta} \mid \mathbf{k} & \sim & \mathrm{Dir}(\boldsymbol{\alpha} + \mathbf{k})
\end{array}
\qquad
\begin{array}{l}
(\sum k_j = n) \\
(\sum \theta_j = 1)
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$$

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\mathbb{E}[\theta_j] = \frac{\alpha_j}{\sum \alpha_i} \qquad \mathbb{V}(\theta_j) = \frac{\mathbb{E}[\theta_j](1 - \mathbb{E}[\theta_j])}{\sum \alpha_i + 1}
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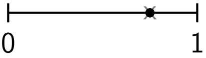
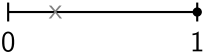
$$y_j^{(1)} = \frac{n^{(0)}}{n^{(0)} + n} \cdot y_j^{(0)} + \frac{n}{n^{(0)} + n} \cdot \frac{k_j}{n} \qquad n^{(1)} = n^{(0)} + n$$

# Simple Example: Dirichlet-Multinomial-Model

Case (i): $y_j^{(0)} = 0.75\,,$ $(n^{(0)} = 8)$ $\quad k_j/n = 0.75$ $(n = 16)$



Case (ii): $y_j^{(0)} = 0.25\,,$ $(n^{(0)} = 8)$ $\quad k_j/n = 1$ $(n = 16)$



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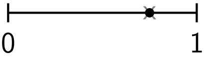

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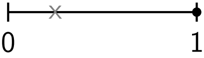
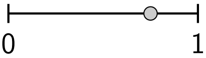


$$\rightarrow \quad \mathbb{E}[\theta_j \mid \mathbf{k}] = y_j^{(1)} = 0.75, \quad \mathbb{V}(\theta_j \mid \mathbf{k}) = 3/400$$



$$(\mathbb{V}(\theta_j) = 1/48)$$

⚠ Posterior inferences do not reflect uncertainty ⚠
due to unexpected observations!

# Conjugate Priors

Weighted average structure is underneath *all common* conjugate priors for exponential family sampling distributions!

$X \overset{iid}{\sim}$ linear, canonical exponential family, i.e.

$$p(x \mid \theta) \propto \exp\left\{\langle \psi, \tau(x) \rangle - n\mathbf{b}(\psi)\right\} \qquad \Big[\psi \text{ transformation of } \theta\Big]$$

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➡ (conjugate) posterior:

$$p(\theta \mid x) \propto \exp\left\{n^{(1)}\left[\langle\psi, y^{(1)}\rangle - \mathbf{b}(\psi)\right]\right\},$$

where $y^{(1)} = \frac{n^{(0)}}{n^{(0)} + n} \cdot y^{(0)} + \frac{n}{n^{(0)} + n} \cdot \frac{1}{n}\tau(x)$ and $n^{(1)} = n^{(0)} + n\,.$

# Bayesian Linear Regression

Are posterior inferences in Bayesian linear regression influenced by prior-data conflict?

$$z_i = x_i^{\mathsf{T}} \beta + \varepsilon_i \quad \Big[ x_i \in \mathbb{R}^p \,,\ \beta \in \mathbb{R}^p \Big]$$

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$$z = \mathbf{X}\beta + \varepsilon \qquad \longrightarrow \quad z \mid \beta, \sigma^2 \sim \mathrm{N}(\mathbf{X}\beta, \sigma^2 \mathbf{I})$$

Prior on $(\beta, \sigma^2)$: generally taken as $p(\beta, \sigma^2) = p(\beta \mid \sigma^2) p(\sigma^2)$.

- SCP: standard conjugate prior model (e.g., O'Hagan 1994)
- CCCP: "canonically constructed conjugate prior"

# Standard Conjugate Prior (SCP)

$$\beta \mid \sigma^2 \sim \mathrm{N}_p(m^{(0)}, \sigma^2 \mathbf{M}^{(0)}) \quad \text{(multivariate Normal)}$$

$$\sigma^2 \sim \mathrm{IG}\,(a^{(0)}, b^{(0)}) \qquad \text{(Inverse Gamma, e.g. } p(\sigma^2) \propto \frac{e^{-\frac{b^{(0)}}{\sigma^2}}}{(\sigma^2)^{a^{(0)}+1}}\text{)}$$

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$$\begin{aligned}
m^{(1)} &= \left(\mathbf{M}^{(0)^{-1}} + \mathbf{X}^\mathsf{T}\mathbf{X}\right)^{-1} \left(\mathbf{M}^{(0)^{-1}} m^{(0)} + \mathbf{X}^\mathsf{T} z\right) \\
\mathbf{M}^{(1)} &= \left(\mathbf{M}^{(0)^{-1}} + \mathbf{X}^\mathsf{T}\mathbf{X}\right)^{-1} \\
a^{(1)} &= a^{(0)} + \frac{n}{2} \\
b^{(1)} &= b^{(0)} + \frac{1}{2}\left(z^\mathsf{T} z + m^{(0)^\mathsf{T}} \mathbf{M}^{(0)^{-1}} m^{(0)} - m^{(1)^\mathsf{T}} \mathbf{M}^{(1)^{-1}} m^{(1)}\right)
\end{aligned}$$

# SCP: Update step for $\beta \mid \sigma^2$

$$\begin{aligned} \mathbb{E}[\beta \mid \sigma^2] &= m^{(0)} \\ \mathbb{E}[\beta \mid \sigma^2, z] &= m^{(1)} = (\mathbf{I} - \mathbf{A})\, m^{(0)} + \mathbf{A}\, \hat{\beta}^{\mathsf{LS}} \\ &\qquad \text{where } \mathbf{A} = \left(\mathbf{M}^{(0)^{-1}} + \mathbf{X}^{\mathsf{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathsf{T}}\mathbf{X} \end{aligned}$$

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Actually of interest:

$$\begin{aligned}
\mathbb{E}[\beta] &= m^{(0)}, & \mathbb{V}(\beta) &= \frac{b^{(0)}}{a^{(0)} - 1}\mathbf{M}^{(0)} = \mathbb{E}[\sigma^2]\,\mathbf{M}^{(0)} \\
\mathbb{E}[\beta \mid z] &= m^{(1)}, & \mathbb{V}(\beta \mid z) &= \frac{b^{(1)}}{a^{(1)} - 1}\mathbf{M}^{(1)} = \mathbb{E}[\sigma^2 \mid z]\,\mathbf{M}^{(1)}
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# SCP: Update step for $\sigma^2$

$$\mathbb{E}[\sigma^2 \mid z] = \frac{2a^{(0)} - 2}{2a^{(0)} + n - 2}\,\mathbb{E}[\sigma^2] + \frac{n-p}{2a^{(0)} + n - 2}\,\hat\sigma^2_{\mathrm{LS}} + \frac{p}{2a^{(0)} + n - 2}\,\hat\sigma^2_{\mathrm{PDC}}$$

$$\begin{aligned}\hat\sigma^2_{\mathrm{LS}} &= \frac{1}{n-p}(z - \mathbf{X}\hat\beta^{\mathrm{LS}})^{\mathsf{T}}(z - \mathbf{X}\hat\beta^{\mathrm{LS}}) \\ \hat\sigma^2_{\mathrm{PDC}} &= \frac{1}{p}(m^{(0)} - \hat\beta^{\mathrm{LS}})^{\mathsf{T}}\Big(\mathbf{M}^{(0)} + (\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}\Big)^{-1}(m^{(0)} - \hat\beta^{\mathrm{LS}})\end{aligned}$$

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Weights:

- $2a^{(0)} - 2$ for $\mathbb{E}[\sigma^2]$: think of $\mathbb{V}(\sigma^2) = \frac{(b^{(0)})^2}{(a^{(0)}-1)^2(a^{(0)}-2)}$
- $n - p$ for $\hat{\sigma}^2_{\mathrm{LS}}$: usual *df*s in least-squares estimate
- $p$ for $\hat{\sigma}^2_{\mathrm{PDC}}$: $\dim(\beta) =$ number of dimensions in which prior-data conflict is possible

# SCP: Summary

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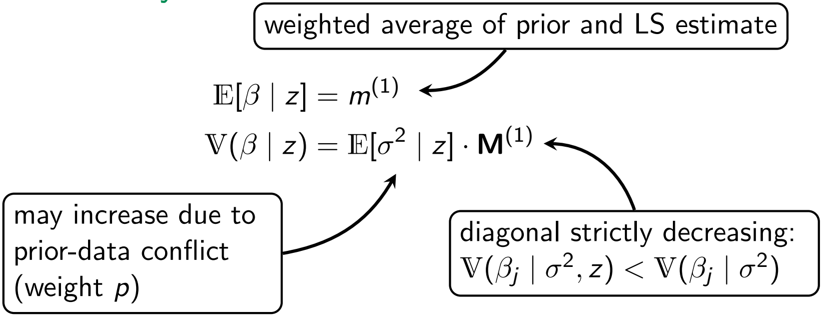

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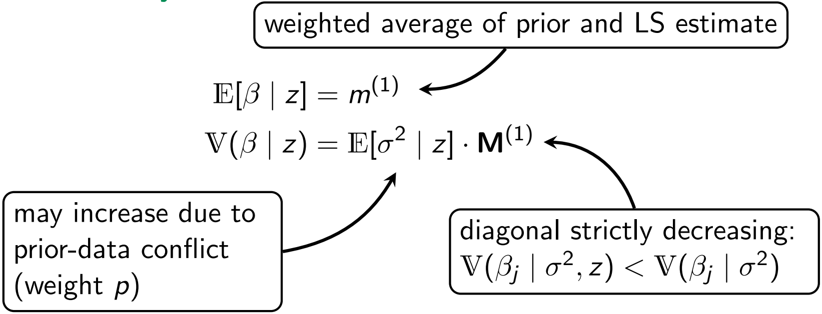

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→ Posterior variance may increase due to prior-data conflict. But:

- Effect of possible increase of $\mathbb{E}[\sigma^2 \mid z]$ contrasted by automatic decrease of $\mathbf{M}^{(1)}$!
- Variance increase *by the same factor* for all $\beta_j$s!

# Canonically Constructed Conjugate Prior (CCCP)

CCCP turns out as a special case of SCP:

$$\beta \mid \sigma^2 \sim \mathrm{N}_p\Big(m^{(0)}\,,\ \sigma^2 \underbrace{\frac{n}{n^{(0)}}\,(\mathbf{X}^\mathsf{T}\mathbf{X})^{-1}}_{=:\mathbf{M}^{(0)}}\Big)$$

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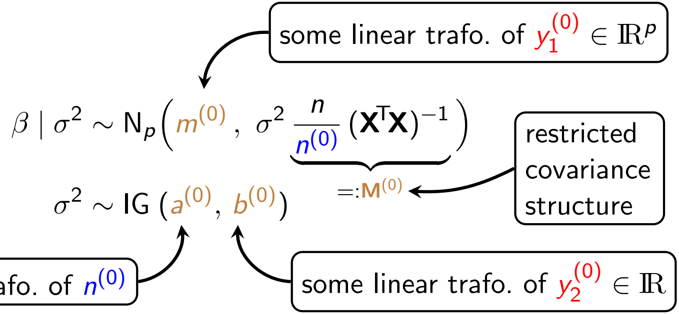

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$$\mathbb{E}[\beta \mid \sigma^2, z] = m^{(1)} = \frac{n^{(0)}}{n^{(0)} + n}\mathbb{E}[\beta \mid \sigma^2] + \frac{n}{n^{(0)} + n}\hat{\beta}^{\mathrm{LS}}$$

$$\mathbb{V}(\beta \mid \sigma^2, z) = \sigma^2\mathbf{M}^{(1)} = \sigma^2\frac{n}{n^{(0)} + n}\,(\mathbf{X}^\mathsf{T}\mathbf{X})^{-1}$$

# CCCP: Update step for $\sigma^2$

Again, for the posterior on $\beta$ it holds that

$$\mathbb{E}[\beta \mid z] = m^{(1)}$$
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$$\mathbb{E}[\sigma^2 \mid z] = \frac{n^{(0)} + p}{n^{(0)} + n + p} \mathbb{E}[\sigma^2] + \frac{n - p}{n^{(0)} + n + p} \hat{\sigma}^2_{\mathrm{LS}} + \frac{p}{n^{(0)} + n + p} \hat{\sigma}^2_{\mathrm{PDC}}$$

$$\hat{\sigma}^2_{\mathrm{PDC}} = \frac{1}{p}(m^{(0)} - \hat{\beta}^{\mathrm{LS}})^{\mathrm{T}} \frac{n^{(0)}}{n^{(0)} + n} \mathbf{X}^{\mathrm{T}}\mathbf{X}(m^{(0)} - \hat{\beta}^{\mathrm{LS}}) \qquad \Big[\mathbb{E}[\,\cdot \mid \sigma^2] = \sigma^2\Big]$$

For CCCP, two other interesting decompositions of $\mathbb{E}[\sigma^2 \mid z]$ exist.

# CCCP: Summary

$$\mathbb{E}[\beta \mid z] = m^{(1)} = \frac{n^{(0)}}{n^{(0)} + n}\mathbb{E}[\beta \mid \sigma^2] + \frac{n}{n^{(0)} + n}\hat{\beta}^{\mathsf{LS}}$$

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# Generalized Bayesian Inference: Basic Idea

Use **set of** priors ➡ base inferences on **set of** posteriors obtained by element-wise updating

# Generalized Bayesian Inference: Basic Idea

Use **set of** priors → base inferences on **set of** posteriors
obtained by element-wise updating
↳ numbers become intervals, e.g.

$$\mathbb{E}[\theta] \quad \rightarrow \quad \left[\underline{\mathbb{E}}[\theta],\, \overline{\mathbb{E}}[\theta]\right] = \left[\min_{p \in \mathcal{M}_\theta} \mathbb{E}_p[\theta],\, \max_{p \in \mathcal{M}_\theta} \mathbb{E}_p[\theta]\right]$$

$$P(\theta \in A) \quad \rightarrow \quad \left[\underline{P}(\theta \in A),\, \overline{P}(\theta \in A)\right] = \left[\min P_p(\theta \in A),\, \max P_p(\theta \in A)\right]$$

# Generalized Bayesian Inference: Basic Idea

Use **set of** priors → base inferences on **set of** posteriors
obtained by element-wise updating
↳ numbers become intervals, e.g.

$$\mathbb{E}[\theta] \quad \rightarrow \quad \left[\underline{\mathbb{E}}[\theta], \overline{\mathbb{E}}[\theta]\right] = \left[\min_{p \in \mathcal{M}_\theta} \mathbb{E}_p[\theta], \max_{p \in \mathcal{M}_\theta} \mathbb{E}_p[\theta]\right]$$

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Shorter intervals ↔ more precise probability statements

**Lottery A**
Number of winning tickets:
exactly known as 5 out of 100
→ $P(\text{win}) = 5/100$

**Lottery B**
Number of winning tickets:
not exactly known, supposedly
between 1 and 7 out of 100
→ $P(\text{win}) = [1/100, 7/100]$

# Generalized Bayesian Inference: Basic Idea

Sets of distributions ↔ Probability Intervals / Expectation Intervals
(„credal sets“) (lower/upper prob.) (lower/upper prevision)

(lower/upper prob.) ↓ Weichselberger (2001)

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# Generalized iLUCK-models

Model for Bayesian inference with sets of priors
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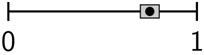
3. set of posteriors $\hat{=}$ set of (element-wise) updated priors
   - ➡ still easy to handle: described as set of $(y^{(1)}, n^{(1)})$'s

$$y^{(1)} = \frac{n^{(0)}}{n^{(0)} + n} \cdot y^{(0)} + \frac{n}{n^{(0)} + n} \cdot \frac{1}{n}\tau(x)$$

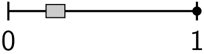
$$n^{(1)} = n^{(0)} + n$$

# Generalized iLUCK-models: Simple Example

Case (i): $y_j^{(0)} \in [0.7, 0.8]\,,$ $(n^{(0)} \in [1, 8])$ $k_j/n = 0.75$ $(n = 16)$



Case (ii): $y_j^{(0)} \in [0.2, 0.3]\,,$ $(n^{(0)} \in [1, 8])$ $k_j/n = 1$ $(n = 16)$



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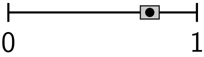
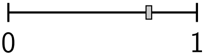


➡ $y_j^{(1)} \in [0.73,\, 0.76]$ $(n^{(0)} \in [17, 24])$



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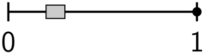

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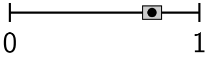

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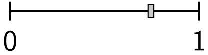

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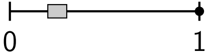

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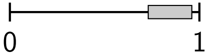

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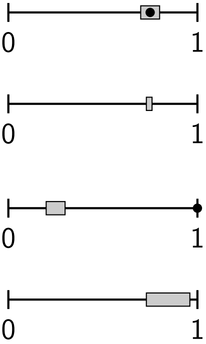

Generalized iLUCK-models lead to cautious inferences if, and only if, caution is needed.

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- Set interval for $\mathbb{E}[\sigma^2]$. (Update step is a bit tricky, though.)

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