Nonparametric Bayesian System Reliability
with Imprecise Prior Information on Component Lifetimes

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UNIVERSITY OF OXFORD

Durham University

DINALOG Dutch Institute for Advanced Logistics

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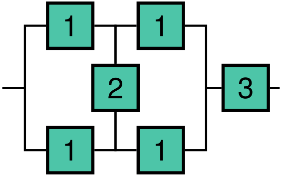
System Reliability



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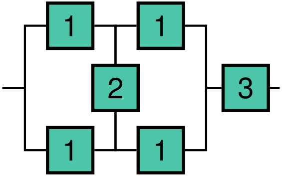

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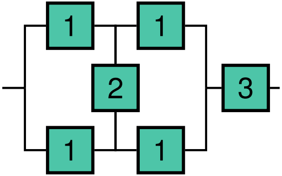

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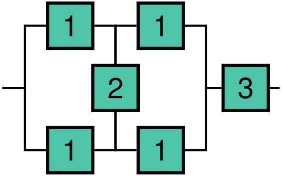

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How to combine these two information sources?

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$f(p)$	$\times$	$f(s \mid p)$	$\propto$	$f(p \mid s)$

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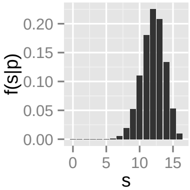
Binomial distribution

$$s \mid p \sim \text{Binomial}(n, p)$$



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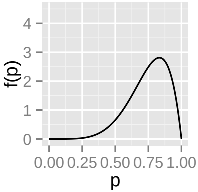

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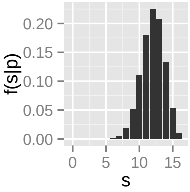

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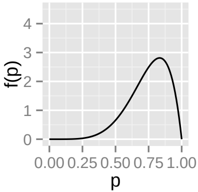

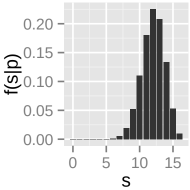

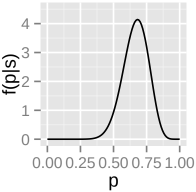

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- conjugate prior makes learning about parameter tractable, just update hyperparameters: $\alpha^{(0)} \rightarrow \alpha^{(n)}$, $\beta^{(0)} \rightarrow \beta^{(n)}$
- closed form for some inferences: $\mathrm{E}[p \mid s] = \frac{\alpha^{(n)}}{\alpha^{(n)}+\beta^{(n)}}$

Nonparametric Component Reliability

Functioning probability p_t^k of $\boxed{\text{k}}$ for each time $t \in \mathcal{T} = \{\dot{t}_1, \dot{t}_2, \ldots\}$

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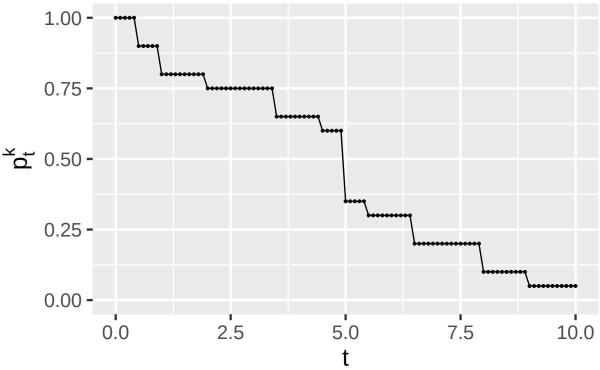

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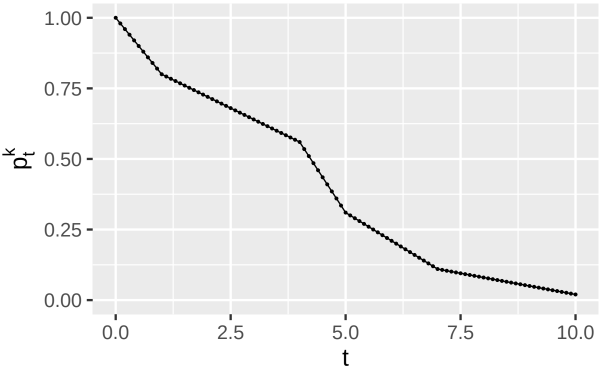

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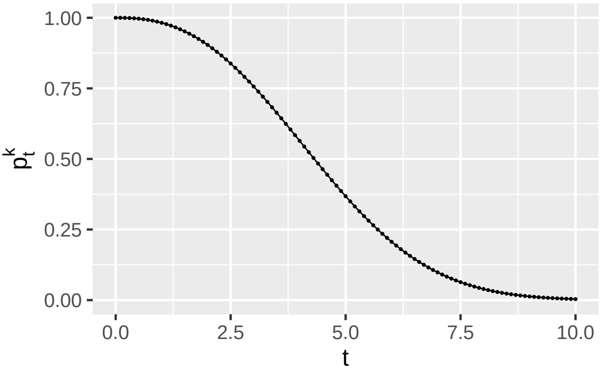

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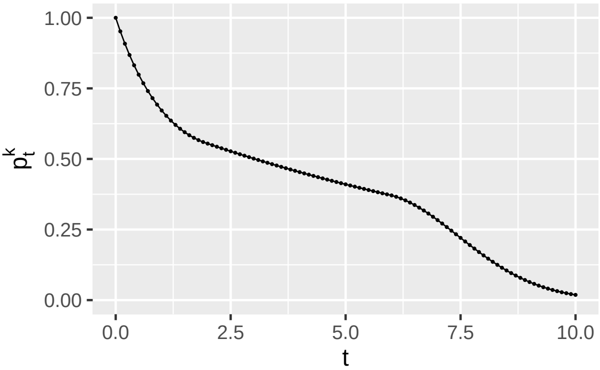

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  Beta posterior for each k and t:

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Prior-Data Conflict

What if expert information and data tell different stories?

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- *informative prior beliefs* and *trusted data* (sampling model correct, no outliers, etc.) are in conflict
- "[...] the prior [places] its mass primarily on distributions in the sampling model for which the observed data is surprising" (Evans and Moshonov 2006)
- there are not enough data to overrule the prior

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$n^{(0)}$ = pseudocounts

$y^{(0)} = \mathrm{E}[p]$

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ML estimator $\hat{p}$

$\mathrm{E}[p \mid s] = y^{(n)}$ is a weighted average of $\mathrm{E}[p]$ and $\hat{p}$!

$\mathrm{Var}[p \mid s] = \dfrac{y^{(n)}(1 - y^{(n)})}{n^{(n)} + 1}$ decreases with n!

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Sets of priors. . .

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Uncertainty about probability statements

smaller sets = more precise probability statements

Lottery A
Number of winning tickets:
exactly known as 5 out of 100
- $P(\text{win}) = 5/100$

Lottery B
Number of winning tickets:
not exactly known, supposedly
between 1 and 7 out of 100
- $P(\text{win}) = [1/100, 7/100]$

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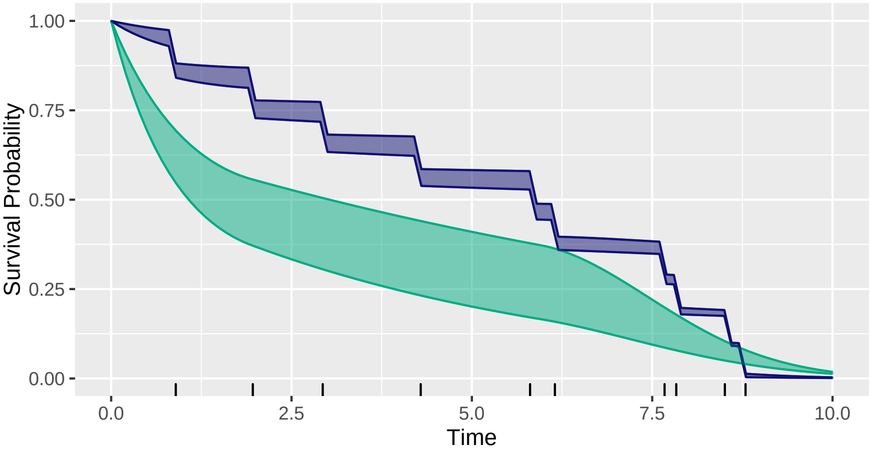
- Bounds for inferences (point estimate, prediction, ...) by min/max over

Component Reliability with Sets of Priors



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$$[\underline{y}^{(0)}, \overline{y}^{(0)}] = \updownarrow$$

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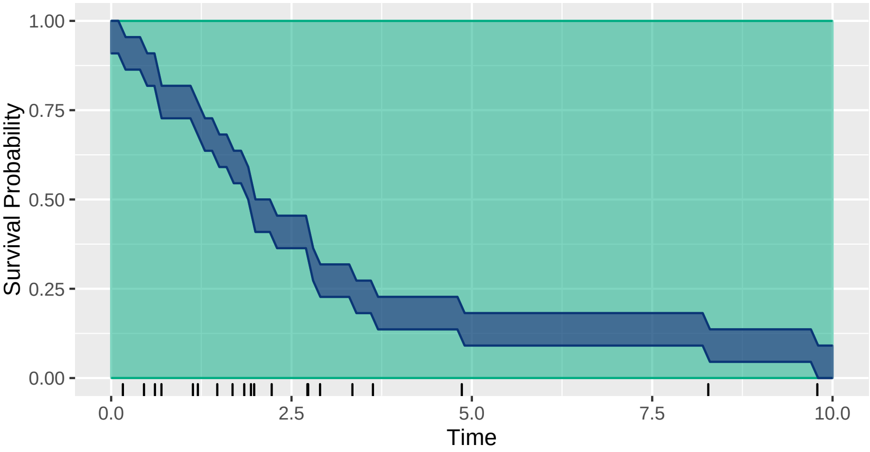

$$[\underline{n}^{(0)}, \overline{n}^{(0)}] = [1, 2]$$
$$[\underline{y}^{(0)}, \overline{y}^{(0)}] = [0, 1]$$

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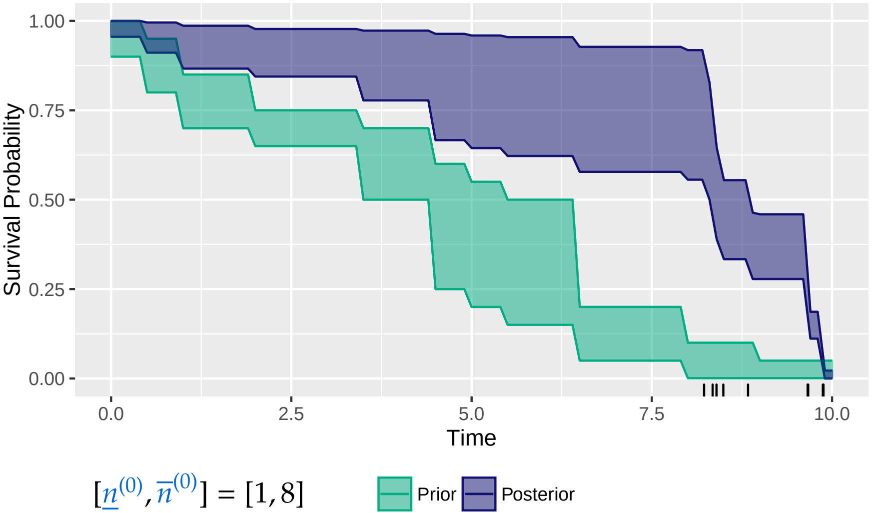

$[\underline{n}^{(0)}, \overline{n}^{(0)}] = [1, 8]$

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System Reliability

- Closed form for the system reliability via the survival signature:

$$R_{\mathsf{sys}}\Big(t \mid \bigcup_{k=1}^{K}\Big\{n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k\Big\}\Big) = P(T_{\mathsf{sys}} > t \mid \cdots)$$

$$= \sum_{l_1=0}^{m_1} \cdots \sum_{l_K=0}^{m_K} \Phi(l_1, \ldots, l_K) \prod_{k=1}^{K} P(C_t^k = l_k \mid n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k)$$

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Survival signature $\Phi(l_1,\ldots,l_K)$
(Coolen and Coolen-Maturi 2012)
$= P$(system functions | $\{l_k$ $\boxed{\text{k}}$'s function$\}^{1:K}$)

l_1	l_2	l_3	Φ
0	0	1	0
1	0	1	0
2	0	1	1/3
3	0	1	1
4	0	1	1

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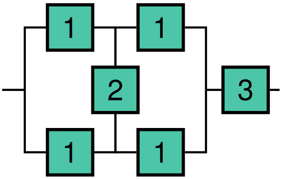

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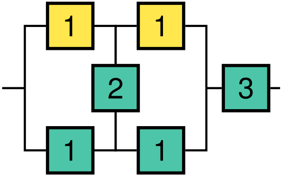
✓



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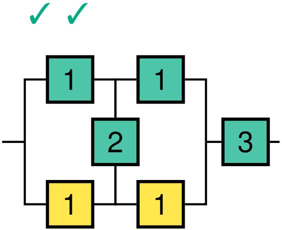

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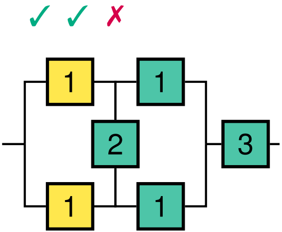

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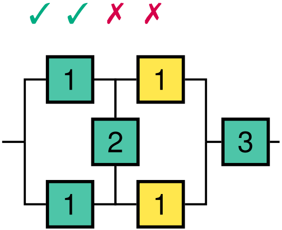

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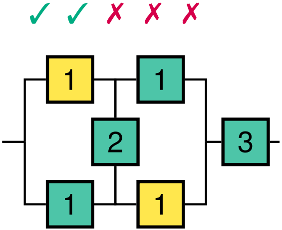

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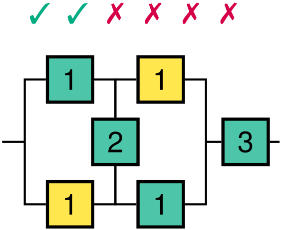

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Posterior predictive probability that in a new system, l_k of the m_k $\boxed{\text{k}}$'s function at time t:

$$\binom{m_k}{l_k} \int [P(T < t \mid p_t^k)]^{l_k} [P(T \geq t \mid p_t^k)]^{m_k - l_k} f(p_t^k \mid n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k)\, dp_t^k$$

- analytical solution for integral: $C_t^k \mid n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k \sim \text{Beta-binomial}$

System Reliability Bounds

- Bounds for $R_{\text{sys}}\Big(t \,\Big|\, \bigcup_{k=1}^{K} \{n_{k,t}^{(0)}, y_{k,t}^{(0)}, \boldsymbol{t}^k\}\Big)$ over $\bigcup_{k=1}^{K} \{ \qquad \}$:
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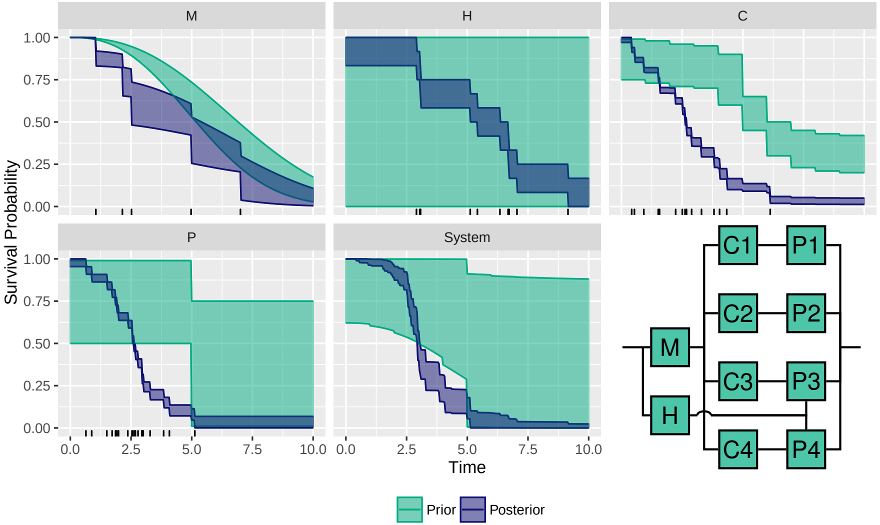

Summary & Outlook

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- Bayesian model combining expert knowledge and test data
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Next steps:

- Allow right-censored observations (RUL estimation)
- Allow dependence between components (common-cause failure, . . .)
- Use for system design (where to put extra redundancy?)
- Use for maintenance planning

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